Problem 1. Let $d_1$ and $d_2$ be metrics on $X$. Show that $\sqrt{d_1}$, $d_1 + d_2$, $\max\{d_1, d_2\}$ are also metrics. Are the functions $\min\{d_1, d_2\}$, $d_1d_2$ metrics on $X$?

Problem 2. Let $(X, d)$ be a metric space. Show that the function

$$D(x, y) = f(d(x, y)) \quad x, y \in X$$

defines a metric if $f$ satisfies the following conditions:

(a) $f(0) = 0$;

(b) $f$ is an increasing function;

(c) $f(x + y) \leq f(x) + f(y)$.

Problem 3. Let $X$ be a vector space, and let $d$ be a metric on $X$ satisfying $d(x, y) = d(x - y, 0)$ and $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for every $x, y \in X$ and every $\alpha \in \mathbb{R}$. Show that $\|x\| = d(x, 0)$ defines a norm on $X$. Give an example of a metric on a vector space $X$ which is not associated with a norm in this way.

Problem 4. Let $(X, d)$ be a metric space.

(a) Show that $|d(x, z) - d(y, z)| \leq d(x, y)$ for all $x, y, z \in X$. Use this inequality to show that if a sequence $(x_n)$ converges to $x$, then the sequence $(d(x_n, y))$ converges to $d(x, y)$ for every $y \in X$.

(b) Show that $|d(x, z) - d(y, u)| \leq d(x, y) + d(z, u)$ for all $x, y, z, u \in X$. Use this inequality to show that if sequences $(x_n)$ and $(y_n)$ converge to $x$ and $y$, respectively, the sequence $(d(x_n, y_n))$ converges to $d(x, y)$.

Problem 5. Let $(X, d_X)$ and $(Y, d_Y)$ be two metric space and let $Z = X \times Y$ be the product metric space with the metric

$$D((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

(a) Show that a sequence $(z_n) = ((x_n, y_n))$ converges in $(Z, D)$ if and only if $(x_n)$ converges in $(X, d_X)$ and $(y_n)$ converges in $(Y, d_Y)$.

(b) Show that a sequence $(z_n) = ((x_n, y_n))$ is Cauchy in $(Z, D)$ if and only if $(x_n)$ is Cauchy in $(X, d_X)$ and $(y_n)$ is Cauchy in $(Y, d_Y)$.

Problem 6. Assume that $(a_n)$ is a Cauchy sequence of real numbers and that $k \geq 2$. Show that also $(a_n^k)$ is Cauchy.

Problem 7. Let $(a_n)$ and $(b_n)$ be sequences such that $|a_n + 1 - a_n| \leq b_n$ for all $n$ and the series $(b_n)$ converges. Show that $(a_n)$ is Cauchy and hence it converges.

Problem 8. Let $I = [1, 3/2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) = -x^2/2 + x + 1$. 
(a) Show that \( f \) is strictly decreasing (i.e., \( f(x) > f(y) \) whenever \( x < y \)) and that \( f(I) \subset I \).

(b) Show that \( |f(x) - f(y)| \leq \frac{1}{2} |x - y| \) for all \( x, y \in I \).

(c) Let \( x_0 = 1 \) and \( x_{n+1} = f(x_n) \) for all \( n \geq 0 \). Use (b) to show that the sequence \( (x_n) \) is a Cauchy and that \( \lim x_n = \sqrt{2} \).

**Problem 9.** Assume that \( 0 < a < b \) and \( x_0 = a, y_0 = b \), and

\[ x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{1}{2} (x_n + y_n), \quad n \geq 0. \]

Show that the sequence \( (x_n) \) is increasing, that \( (y_n) \) is decreasing, and that both converge to the same limit (you don’t have to find the limit).

**Problem 10.** Discuss the convergence or divergence of the following series:

(a) \( \sum \left( \frac{\sqrt{n+1} - \sqrt{n}}{n} \right) \)

(b) \( \sum \frac{1}{(\ln n)^p} \)

(c) \( \sum \frac{1}{n(\ln n)^p} \)

(d) \( \sum \frac{1}{(\ln n)^m} \)

(e) \( \sum (-1)^{n+1} \frac{\ln n}{n} \)

(f) \( \sum n^{1/n} a_n \) where \( \sum a_n \) is a convergent series.

(g) \( \sum a_n e^{-nx} \) where \( x > 0 \) and the partial sums of the series \( \sum a_n \) are bounded.