Submit solutions to all of the problems. Each problem is worth the same number of points. Collaboration is allowed, but you need to turn in individually written solutions.

**Problem 1.** Let \( f : (X, d) \to (Y, \rho) \). Show that the following are equivalent:

(a) \( f \) is continuous.

(b) \( f(A) \subset f(A) \) for every \( A \subset X \).

(c) \( f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ \) for every \( B \subset Y \).

**Problem 2.**

(a) Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in a metric space \((X, d)\). Assume that \( \{x_n\} \) is Cauchy and that \( d(x_n, y_n) \to 0 \). Show that \( \{y_n\} \) is Cauchy in \((X, d)\).

(b) A subset \( D \) of \( X \) is called dense if \( D = X \). Suppose that every Cauchy sequence of points in \( D \) converges to some point of \( X \). Show that \( (X, d) \) is complete.

**Problem 3.**

(a) Consider \( \mathbb{R} \) with the metric
\[
\rho(x, y) = |\arctan x - \arctan y|.
\]
Is \((\mathbb{R}, \tau)\) a complete metric space?

(b) Consider \( \mathbb{R} \) with the metric
\[
\tau(x, y) = |x^3 - y^3|, \quad x, y \in \mathbb{R}.
\]
Is \((\mathbb{R}, \tau)\) a complete metric space?

**Problem 4.**

(a) Let \((X, \|\cdot\|)\) be a normed space and let \((x_n)\) be a sequence in \( X \) satisfying
\[
\sum_{n=1}^{\infty} \|x_n\| < \infty.
\]
Show that \((x_n)\) is a Cauchy sequence.

(b) Let \((X, \|\cdot\|)\) be a normed space. Show that \((X, \|\cdot\|)\) is complete if and only if \( \overline{B}_1(0) = \{x \in X \mid \|x\| \leq 1\} \) is complete.

(c) Let \( S \) be the vector space of all real sequences \( x = (x_n) \) such that \( x_n = 0 \) for all but finitely many \( n \). Show that \( S \) is not complete with respect to the norm \( \|x\| = \sup_n |x_n| \).