MATH 251
Final Exam (Exam 2)
August 14, 2009

ANSWER KEY

1. (a) \( Y = (A t^2 + B t) \cos(2t) + (C t^2 + D t) \sin(2t) \)
   (b) \( Y = (A t^2 + B t) e^{3t} \cos(2t) + (C t^2 + D t) e^{3t} \sin(2t) \)
   (c) \( Y = A + (B t^2 + C t) e^t \)

2. (a) \( \mu(t) = e^{-2t} \)
   (b) \( y = -e^{-t} + (y_0 + 1) e^{2t} \)
   (c) when \( y_0 = -1 \) (limit = 0)

3. \( y = e^{-t} \)

4. (a) \( F(s) = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^3} \)
   (b) \( f(t) = 3u_3(t) e^{2t} - 6 \cos(3t - 9) \)

5. \( y = 2e^{-t} - e^{-2t} \)

6. \( x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t} \)

7. (a) saddle point, unstable
   (b) spiral point, asymptotically stable (i.e., a spiral sink)
   (c) node, asymptotically stable (i.e., a nodal sink)

8. (a) \( F(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & x = 0, 1 \\ -x^2, & -1 < x < 0 \end{cases} \)
   \( F(x + 2) = F(x) \)

   (b) \( a_0 = a_n = 0, \quad b_n = 2 \int_0^1 x^2 \sin(n \pi x) \, dx, \quad n = 1, 2, 3, \ldots \)

9. \( u(x, t) = 2e^{-4e^t} \sin(\pi x) - 4e^{-16e^t} \sin(2 \pi x) \)

The question states that the bar has insulated ends. But the actual boundary conditions tell that the ends are both kept at constant zero degree. The solution above is based on the given boundary conditions. The solution for the problem with insulated ends, and the given initial condition, would be much more complex and difficult (it would be quite infeasible to be included in an exam).