This exam has 16 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.

Do not write in this box.

1
through
11: ______ (66)
12: ______ (16)
13: ______ (20)
14: ______ (14)
15: ______ (16)
16: ______ (18)
Total: ______
1. (6 points) Consider the nonlinear differential equation
\[ y' = \sin(\pi y) \]
Which of following statements is false?

(a) The equilibrium solution \( y(t) = -1 \) is asymptotically stable.
(b) The equation is autonomous and separable.
(c) The general solution is \( y(t) = -\frac{1}{\pi} \cos(\pi t) + C \).
(d) If \( y(0) = 1/3 \), then \( \lim_{t \to \infty} y(t) = 1 \).

2. (6 points) Consider the initial/boundary value problems below. Which is certain to have a unique solution for every value of \( \alpha \)?

- \( \checkmark \) I \( y'' + 4y = t^2 e^{-6t}, \ y(\alpha) = \alpha, \ y'(\alpha) = -\alpha. \)
- \( \times \) II \( y'' + 4y = 0, \ y(0) = 0, \ y'(\alpha) = 0. \)
- \( \times \) III \( t^2 y' + 4y = 0, \ y(-\alpha) = 0. \)

(a) I only.
(b) II only.
(c) III only.
(d) I, II, and III.
3. (6 points) Let \( y_1(t) \) and \( y_2(t) \) be two solutions of the second order linear equation

\[ t^2 y'' - 2ty' + \cos(t)y = 0. \]

If the Wronskian \( W(y_1, y_2)(1) = 3 \), find \( W(y_1, y_2)(2) \).

(a) \(-\frac{3}{2}\)

(b) \(\frac{3}{4}\)

(c) 12

(d) -6

\[
\begin{align*}
\rho(t) &= -\frac{2t}{t^2} = -\frac{2}{t} \\
W &= C \cdot \exp \int -\rho(t) \, dt \\
&= C \cdot \exp \int \frac{2}{t} \, dt \\
&= C \cdot \exp (\ln t^2) = C \cdot t^2 \\
C \cdot 1 &= 3 \rightarrow C = 3 \\
W(2) &= 3 \cdot 4 = 12.
\end{align*}
\]

4. (6 points) Which statements below regarding the following linear equation are true?

\[ y^{(4)} + 8y'' + 16y = 0. \]

\[ r^4 + 8r^2 + 16 = 0, \quad (r^2 + 4)^2 = 0. \]

\( r_{1,2} = \pm 2i, \quad r_{3,4} = -2i \)

\( \checkmark \) I \quad \text{y = } C_1 \cos(2t) + C_2 \sin(2t) + C_3 t \cos(2t) + C_4 t \sin(2t) \text{ is its general solution.}

\( \checkmark \) II \quad \text{Given } y(t_0) = 0, \quad y'(t_0) = 1, \quad y''(t_0) = 2, \quad y'''(t_0) = 3, \text{ there is a unique solution for any } t_0.

\( \checkmark \) III \quad \text{The trivial solution } y = 0 \text{ satisfies the equation.}

(a) II only.

(b) III only.

(c) I and III.

(d) I, II, and III.
5. (6 points) Which of the following equations describes a mass-spring system that is undergoing resonance?

(a) \( y'' + 6y' + 9y = -2 \cos(3t) \)

(b) \( 2y'' + 50y = 7 \sin(5t) \)

(c) \( y'' - 4y = 12 \cos(2t) \)

(d) \( 3y'' + 12y = 5 \sin(4t) \)

6. (6 points) Find the Laplace transform of \( f(t) = u_4(t)te^{3t} \).

(a) \( F(s) = e^{-4s} \frac{1}{s(s - 3)^2} \)

(b) \( F(s) = e^{-4s + 12} \frac{4s - 11}{(s - 3)^2} \)

(c) \( F(s) = e^{-4s - 12} \frac{4s + 13}{(s - 3)^2} \)

(d) \( F(s) = e^{-4s} \frac{1}{(s - 3)^2} \)
7. (6 points) Find the inverse Laplace transform of

\[
F(s) = e^{-2\pi s} \frac{4s}{s^2 + 6s + 25} \quad \text{or} \quad G(s) = \frac{4(s + 3) - 3.4}{(s + 3)^2 + 4^2}
\]

(a) \(f(t) = u_{2\pi}(t) (4e^{-3t+6\pi} \cos(4t) - 3e^{-3t+6\pi} \sin(4t))\)

(b) \(f(t) = u_{2\pi}(t) (4e^{-3t} \cos(4t - 8\pi) - 3e^{-3t} \sin(4t - 8\pi))\)

(c) \(f(t) = 4u_{2\pi}e^{-3t} \cos(4t)\)

(d) \(f(t) = 4u_{2}(t)e^{-3t+6\pi} \cos(4t - 8\pi)\)

\[g(t) = e^{-3t} \left[ 4 \cos 2t - 3 \sin t \right]\]

\[\int_{-3t+6\pi}^{0} = u_{2\pi}(t) e^{-3t} \left[ 4 \cos 2(t-x) - 3 \sin x \right]\]

8. (6 points) Which of the following systems of linear equations has the property that some, but not all, of its nonzero solutions converge to its critical point at (0, 0) as \(t \to +\infty\)?

(a) \(x' = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} x\)

(b) \(x' = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} x\)

(c) \(x' = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} x\)

(d) \(x' = \begin{bmatrix} -2 & 3 \\ 0 & -3 \end{bmatrix} x\)

Compute eigenvalues for each case, and find the saddle point. I skip the details.
9. (6 points) Given that (3, 1) is a critical point of the nonlinear system

\[ \begin{align*}
    x' &= 3xy + 3y - x^2 - x \\
    y' &= y^2 + x - y - xy
\end{align*} \]

The critical point (3, 1) is an

(a) unstable improper node.
(b) unstable saddle point.
(c) asymptotically stable spiral point.
(d) asymptotically stable node.

\[ \mathbf{J}(x, y) = \begin{pmatrix} 3y - 2x - 1 & 3x + 3 \\
                        1 - y & 2y - 1 - x \end{pmatrix} \]

\[ \mathbf{J}(3, 1) = \begin{pmatrix} -4 & 12 \\
                                  0 & -2 \end{pmatrix} \]

\[ \lambda_1 = -4 \]
\[ \lambda_2 = -2 \]

10. (6 points) Using the substitution \( u(x, y) = X(x)Y(y) \), where \( u(x, y) \) is not the trivial solution, consider the two statements below.

\( \sqrt{1} \) The equation \( u_{xx} = \frac{1}{x} u_x + \frac{1}{x^2} u_{yy} \) can be separated into two ordinary differential equations.

\( \times \) The boundary conditions \( u(0, y) = \pi \) and \( u_x(2\pi, y) = 0 \) can be rewritten into \( X(0) = \pi \) and \( X'(2\pi) = 0 \).

What can you say regarding the truthfulness of these statements?

(a) Only (1) is true.
(b) Only (2) is true.
(c) Both are true.
(d) Neither is true.
11. (6 points) Which function(s) below can be represented by a Fourier cosine series?

- **I** \( f(x) = x^4, \quad -\infty < x < \infty. \)
- **II** \( g(x) = 3x^2 - 9, \quad -10 < x < 10, \quad g(x + 20) = g(x). \)
- **III** \( h(x) = \sin^2(x) + \cos^3(2x), \quad -\infty < x < \infty. \)

(a) II only.
(b) I and III.
(c) II and III.
(d) I, II, and III.
12. (16 points) Consider the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(4) = 0. \]

(a) (12 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem. You must show all supporting work to your answer.

\[ \lambda > 0, \text{ let } \lambda = k^2. \]

\[ X(x) = C_1 \cos kt + C_2 \sin kt. \]

\[ X'(x) = -k C_1 \sin kt + k C_2 \cos kt. \]

\[ X(0) = C_1 = 0, \quad X'(4) = k C_2 \cos 4k = 0. \]

Thus, we have \( C_2 \neq 0 \) and must have \( \cos 4k = 0 \) which implies

\[ 4k = n\pi - \frac{\pi}{2}, \quad k = \frac{n - \frac{1}{2}}{4} \pi, \quad n = 1, 2, \ldots \]

\[ \left\{ \begin{array}{l}
\lambda_n = k_n^2 = \left[ \frac{n - \frac{1}{2}}{4} \pi \right]^2, \\
X_n = \sin \left( \frac{n - \frac{1}{2}}{4} \pi \right) x.
\end{array} \right. \quad n = 1, 2, \ldots \]

(b) (4 points) Is \( \lambda = 0 \) an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

If \( \lambda = 0, \quad X'' = 0, \quad X(x) = ax + b, \quad X'(x) = a. \)

B.c.s give \( X(0) = b = 0 \)

\[ \Rightarrow X(x) = 0, \text{ trivial solution.} \]

\[ \Rightarrow \lambda = 0 \text{ is not an eigenvalue.} \]
13. (20 points) Let \( f(x) = \begin{cases} x^3, & 0 \leq x < 1 \\ 1, & 1 < x < 2 \end{cases} \).

(a) (4 points) Consider the odd periodic extension, of period \( T = 4 \), of \( f(x) \). Sketch 3 periods, on the interval \(-6 < x < 6\), of this odd periodic extension.

(b) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -3 \)? At \( x = 6 \)?

\[
\begin{align*}
\mathcal{A}+ & \quad x = -3 \\
& \quad \rightarrow 1 \\
\mathcal{A}+ & \quad x = 6 \\
& \quad \rightarrow 0
\end{align*}
\]

(c) (4 points) Consider the even periodic extension, of period \( T = 4 \), of \( f(x) \). Sketch 3 periods, on the interval \(-6 < x < 6\), of this even periodic extension.

(d) (3 points) Find \( \frac{a_0}{2} \), the constant term of the Fourier series of the even periodic function described in (c).

\[
\frac{a_0}{2} = \frac{1}{2} \int_{0}^{2} f(x) \, dx = \frac{1}{2} \left[ \int_{0}^{1} x^3 \, dx + \int_{1}^{2} \, dx \right]
\]

\[
= \frac{1}{2} \left[ \frac{1}{4} + 1 \right] = \frac{5}{8}
\]

(e) (5 points) Which of the integrals below can be used to find the Fourier cosine coefficients of the even periodic extension in (c)?

(i) \[ a_n = \left( \int_{0}^{1} x^3 \cos \frac{n\pi x}{2} \, dx + \int_{1}^{2} \cos \frac{n\pi x}{2} \, dx \right) \]

(ii) \[ a_n = \frac{1}{2} \left( \int_{0}^{1} x^3 \cos \frac{n\pi x}{2} \, dx + \int_{1}^{2} \cos \frac{n\pi x}{2} \, dx \right) \]

(iii) \[ a_n = \frac{1}{2} \left( \int_{-2}^{-1} \cos \frac{n\pi x}{2} \, dx + \int_{-1}^{1} x^3 \cos \frac{n\pi x}{2} \, dx + \int_{1}^{2} \cos \frac{n\pi x}{2} \, dx \right) \]

(iv) \[ a_n = -\left( \int_{-2}^{-1} \cos \frac{n\pi x}{2} \, dx + \int_{-1}^{1} x^3 \cos \frac{n\pi x}{2} \, dx + \int_{1}^{2} \cos \frac{n\pi x}{2} \, dx \right) \]
14. (14 points) Consider the following heat conduction initial-boundary value problems

I. \[ 12u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0 \]
   \[ u(0, t) = 0, \quad u(4, t) = 0, \]
   \[ u(x, 0) = 50 - 10 \cos(6\pi x). \]

II. \[ 12u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0 \]
    \[ u(0, t) = 30, \quad u(4, t) = 30, \]
    \[ u(x, 0) = 50 - 10 \cos(6\pi x). \]

III. \[ 12u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0 \]
    \[ u_x(0, t) = 0, \quad u_x(4, t) = 0, \]
    \[ u(x, 0) = 50 - 10 \cos(6\pi x). \]

(a) (3 points) Which problem (I, II, or III) models the temperature distribution of a rod with both ends perfectly insulated?

   III

(b) (3 points) What is the physical meaning of the initial condition of each heat conduction problem?

   Initial temperature distribution

(c) (5 points) Let \( u_1(x, t), u_2(x, t), \) and \( u_3(x, t) \) be the solution of problems I, II, and III, respectively. Consider the temperatures at the middle of the rod (that is, at \( x = 2 \)) as \( t \to \infty \) of each solution. Which has the highest limiting temperature?

   (i) I \[ \text{steady state:} \quad U(x) = 0, \quad U(2) = 0 \]
   (ii) II \[ U(x) = 30, \quad U(2) = 30 \]
   (iii) III \[ U(x) = \text{average of } U(x, 0) = 50, \quad U(2) = 50 \]
   (iv) The limiting temperatures will be the same.

(d) (3 points) Suppose the initial condition is, instead, \( u(x, 0) = 25 + 25\cos(3\pi x) \). Will your answer to part (c) change? Why or why not?

   For III, \[ U(x) = 25, \quad U(2) = 25. \]
   Yes, answer should change to (iv).
15. (16 points) Suppose the displacement \( u(x, t) \) of a piece of flexible string that has both ends firmly fixed in places is given by the initial-boundary value problem

\[
25u_{xx} = u_{tt}, \quad 0 < x < 6, \quad t > 0
\]

\[
u(0, t) = 0, \quad u(6, t) = 0,
\]

\[
u(x, 0) = 6x - x^2,
\]

\[
u_t(x, 0) = 0.
\]

(a) (2 points) What is the propagation speed of the standing waves?

\[
C^2 = 25, \quad C = 5
\]

(b) (3 points) TRUE or FALSE: At \( t = 0 \) the string has zero initial velocity.

\[
\text{F} \quad \text{blc} \quad \nu_t(x, 0) = 0
\]

(c) (5 points) In what specific form will its general solution appear?

\[
(1) \quad u(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{5n\pi t}{6} \cos \frac{n\pi x}{6},
\]

\[
(2) \quad u(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{5n\pi t}{6} \sin \frac{n\pi x}{6},
\]

\[
(3) \quad u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{5n\pi t}{6} \cos \frac{n\pi x}{6},
\]

\[
(4) \quad u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{5n\pi t}{6} \sin \frac{n\pi x}{6}
\]

(d) (3 points) TRUE or FALSE: The coefficients of the solution in part (c) above can be found using the integral

\[
C_n = \frac{1}{L} \int_0^L (6x - x^2) \sin \frac{n\pi x}{6} \, dx.
\]

\[
C_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{6} \, dx \quad \text{F}
\]

(e) (3 points) TRUE or FALSE: The string will NEVER reach its steady-state displacement.

\[
\text{T}
\]
16. (18 points) Suppose the temperature distribution function \( u(x, t) \) of a rod that has both ends kept at constant temperatures is given by the initial-boundary value problem

\[
\begin{align*}
4u_{xx} &= u_t, & 0 < x < \pi, & t > 0 \\
u(0, t) &= 0, & u(\pi, t) &= 10\pi \\
u(x, 0) &= 10x + \sin(x) + 5\sin(3x) - 0.5\sin(9x).
\end{align*}
\]

(a) (3 points) What is the steady-state solution of this problem?

\[
U(x) = 10x
\]

(b) (10 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

\[
U(x, t) = 10x + \sum_{n=1}^{\infty} C_n \cdot e^{-\left(\frac{2n}{\pi}\right)^2 t} \sin nx
\]

I.e. \( U(x, 0) = 10x + f(x) \)

Fit it in: \[\sum_{n=1}^{\infty} C_n \cdot \sin nx = 10x + 5\sin 3x - 0.5\sin 9x\]

\[\Rightarrow \quad c_1 = 1, \quad c_2 = 5, \quad c_7 = -0.5, \quad \text{All other } c_n = 0\]

Solution: \[
U(x, t) = 10x + e^{-\frac{36}{\pi^2} t} \cdot 5 \cdot e^{-\frac{36}{9^2} t} \sin 3x - 0.5 \cdot e^{-\frac{36}{9^2} t} \sin 9x
\]

(c) (2 points) What is \( \lim_{t \to \infty} u(1, t) \)?

\[
\lim_{t \to \infty} U(1) = 10
\]

(d) (3 points) Suppose the boundary conditions are, instead, \( u(0, t) = 0 \) and \( u(\pi, t) = \pi \). What is \( \lim_{t \to \infty} u(1, t) \)? (Hint: you do not need to re-solve the entire problem.)

New steady state: \( \overline{U_1}(x) = x \).

\[
\lim_{t \to \infty} U(1, t) = \overline{U}(1) = 1.
\]