This exam has 16 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.**
1. (6 points) Which initial or boundary value problem below is guaranteed to have a unique solution according to the Existence and Uniqueness theorems?

(a) $t^2 y'' + ty' + e^{2t}y = 2,$ \hspace{1cm} y(2) = \sqrt{3}, \hspace{1cm} y'(2) = \frac{1}{e}.
(b) $y'' + \sin(t)y' + 10ty = \ln(t),$ \hspace{1cm} y(-5) = 1, \hspace{1cm} y'(-5) = 4.
(c) $y' + \tan(t)y = \sec(t),$ \hspace{1cm} y'\left(\frac{3\pi}{2}\right) = 9.
(d) $y'' - t^3y' + e^{-3t}y = 0,$ \hspace{1cm} y'(-1) = -1, \hspace{1cm} y'(\pi) = 1.

2. (6 points) Which of the equations below is an exact equation whose general solution is $2xy^3 + e^x y^2 = C$?

(a) $(e^x y^3 + e^x y^2) + \left(\frac{xy^4}{2} + \frac{e^x y^3}{3}\right)y' = 0$
(b) $(2y^3 + e^x y^2) + (6xy^2 + 2e^x y)y' = 0$
(c) $(\frac{xy^4}{2} + \frac{e^x y^3}{3}) + (x^2 y^3 + e^x y^2)y' = 0$
(d) $(6xy^2 + 2e^x y) + (2y^3 + e^x y^2)y' = 0$
3. (6 points) Which of the functions below is a solution of the nonhomogeneous linear equation

\[ y'' + y' - 6y = 2\cos(2t) - 10\sin(2t) ? \]

(a) \( y = e^{2t} - e^{-3t} \)
(b) \( y = 2\cos(2t) \)
(c) \( y = 3e^{-3t} + 2\cos(2t) - 10\sin(2t) \)
(d) \( y = \sin(2t) - 6e^{2t} \)

4. (6 points) Let \( y_1(t) \) and \( y_2(t) \) be any two solutions of the second order linear equation

\[ (1 + i^2)y'' + 2ty' - i^2 \tan(t)y = 0. \]

What is the general form of their Wronskian, \( W(y_1, y_2)(t) \)?

(a) \( \frac{C}{1 + i^2} \)
(b) \( C\arctan(t) \)
(c) \( Ce^{-\arctan(t)} \)
(d) \( C(1 + i^2) \)
5. (6 points) Suppose the Laplace transform of $f(t)$ is $F(s) = \frac{s}{s^2 + 1}$. What is the Laplace transform $\mathcal{L}\{te^t f(t)\}$?

(a) $\frac{s - 1}{s^2 + 1}$

(b) $\frac{-s^2 + 2s}{(s^2 - 2s + 2)^2}$

(c) $\frac{-s^2 + 1}{(s^2 + 1)^2}$

(d) $\frac{s^2 - 2s}{(s^2 + 2s + 2)^2}$

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6. (6 points) Find the inverse Laplace transform of

$$F(s) = \frac{2e^{-s}}{s^2 + 4s + 5}.$$ 

(a) $f(t) = 2\delta(t - 1)e^{-2t} \sin(t)$

(b) $f(t) = u_1(t)e^{-2t-2} \sin(t+1)$

(c) $f(t) = u_1(t)e^{-t+1} \sin(2t-2)$

(d) $f(t) = 2u_1(t)e^{-2t+2} \sin(t-1)$
7. (6 points) Determine the value of $\alpha$ so that the following system of linear equations has a critical point $(0,0)$ that is an asymptotically stable improper node.

$$x' = \begin{pmatrix} \alpha & 2 \\ -1 & 0 \end{pmatrix} x$$

(a) $-1$
(b) $-\sqrt{2}$
(c) $2\sqrt{2}$
(d) $-2\sqrt{2}$

8. (6 points) Given that $(-1,2)$ is a critical point of the nonlinear system

$$x' = (x - 2y)(x + 1)$$
$$y' = (y - 2)(x + y)$$

The critical point $(-1,2)$ is an

(a) unstable improper node.
(b) asymptotically stable node.
(c) unstable saddle point.
(d) asymptotically stable spiral point.
9. (6 points) Using the substitution \( u(x, t) = X(x)T(t) \), where \( u(x, t) \) is not the trivial solution, consider the two statements below.

(1) The equation \( u_{xt} = u_{tt} + 4u_t \) can be separated into two ordinary differential equations.

(2) The boundary conditions \( u_x(0, t) = 0 \) and \( u(4, t) = 0 \) can be rewritten into \( X'(0) = 0 \) and \( X(4) = 0 \).

What can you say regarding the truthfulness of these statements?

(a) Only (1) is true.
(b) Only (2) is true.
(c) Both are true.
(d) Neither is true.

10. (6 points) Find the steady-state solution, \( v(x) \), of the heat conduction problem with nonhomogeneous boundary conditions:

\[
5u_{xx} = u_t, \quad 0 < x < 10, \quad t > 0 \\
u(0, t) + u_x(0, t) = 0, \quad u(10, t) = 18, \\
u(x, 0) = f(x) = x.
\]

(a) \( v(x) = \frac{9}{5}x + 18 \)
(b) \( v(x) = 2x - 2 \)
(c) \( v(x) = 3x - 12 \)
(d) \( v(x) = x + 8 \)
11. (6 points) Find the Fourier sine coefficient corresponding to \( n = 5, \ b_n \), of the Fourier series representing the function

\[ f(x) = x, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x). \]

(a) \( b_5 = \frac{2}{5} \)
(b) \( b_5 = -\frac{2}{5} \)
(c) \( b_5 = \frac{2}{25} \)
(d) \( b_5 = -\frac{2}{25} \)

(1 pt if T/F is correctly stated.

2 pts for the correct reason)

12. (12 points) True or false (you need to briefly explain each answer):

(a) (3 points) The first order equation \( y' - y^2 = 4 \) is separable but not linear.

**TRUE**

It is separable because we can rewrite the equation as \( \frac{dy}{dt} = 4 + y^2 \).

It is not linear because of the \( y^2 \) term.

(b) (3 points) The mass-spring system described by the equation \( u'' + 100u = 5 \cos(10t) \) exhibits resonance.

**TRUE**

\( \omega = 10 \) and \( \omega_0 = \sqrt{k/m} = \sqrt{100} = 10 \).

Since \( \omega = \omega_0 \), it is the resonance system.

(c) (3 points) The equation \( u'' - 5u' + 6u = 0 \) describes the behavior of a certain overdamped mass-spring system.

**FALSE**

\( \delta = -5 < 0 \).

So the equation does not describe a mass spring system.

(d) (3 points) Any linear combination of the functions \( y_1(t) = e^{2t}, y_2(t) = 2e^t \), and \( y_3(t) = -3 \) will be another solution of the equation \( y'' = 3y'' - 2y' \).

**TRUE**

Char eq: \( r^2 - 3r + 2 = 0 \) \( \Rightarrow r(r-2)(r-1) = 0 \) \( \Rightarrow r = 0, 1, 2 \).

So \( e^0, e^t, e^{2t} \) form the fundamental set of solution.
13. (16 points) Consider the two-point boundary value problem

\[ X'' + AX = 0, \quad X'(0) = 0, \quad X(1) = 0. \]

(a) (12 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem.

Let \( \lambda = \mu^2 \) where \( \mu \neq 0 \).
The equation becomes \( X'' + \mu^2 X = 0 \).
Its char. eq is \( 0 = \mu^2 + \mu^2 = \mu = \pm \mu i \).
So \( X = c_1 \cos \mu x + c_2 \sin \mu x \). \hspace{1cm} (4 pts)

Note that \( X' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x \).

Invoking boundary conditions,
\[ X(0) = 0; \quad 0 = c_2 \mu \quad \Rightarrow \quad c_2 = 0. \quad \text{So now} \quad X = c_1 \cos \mu x. \quad (2 pts \ for \ c_2 = 0) \]
\[ X(1) = 0; \quad 0 = c_1 \cos \mu \quad \Rightarrow \quad c_1 = 0 \ or \ \cos \mu = 0. \]

We're looking for nontrivial solutions, we need to assume \( c_1 \neq 0 \).
So \( \cos \mu = 0 \ \Rightarrow \ \mu_n = (\frac{(2n-1)\pi}{2}, \ n = 1, 2, 3, \ldots) \quad (2 pts \ for \ \mu) \)

So the eigenvalues are \( \lambda_n = \mu_n^2 = (\frac{(2n-1)\pi}{2})^2, \ n = 1, 2, 3, \ldots \) \hspace{1cm} (2 pts for \( \lambda_n \))

and the corresponding eigenfunctions are \( X_n(x) = \cos \mu_n x = \cos(\frac{(2n-1)\pi x}{2}), \ n = 1, 2, 3, \ldots \)

(b) (4 points) Is \( \lambda = 0 \) an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

Let \( \lambda = 0 \) . Then \( X'' = 0 \ \Rightarrow \ X = c_1 x + c_2. \)

Note that \( X' = c_1. \) \hspace{1cm} (1 pt for \( X \))

Apply boundary conditions,
\[ X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X = c_2 \quad (1 pt \ for \ c_1) \]
\[ X(1) = 0 \Rightarrow c_2 = 0 \quad (1 pt \ for \ c_2) \]
So \( X = 0 \), which is a trivial solution. So \( \lambda = 0 \) is not an eigenvalue. \hspace{1cm} (1 pt)
14. (20 points) Let \( f(x) = 2 - x, \quad 0 < x < 2. \)

(a) (4 points) Consider the odd periodic extension, of period \( T = 4, \) of \( f(x). \) Sketch 3 periods, on the interval \(-6 < x < 6, \) of this odd periodic extension.

(b) (3 points) TRUE or FALSE: The Fourier sine coefficients of the odd periodic extension in (a) can be found by

\[
b_n = \frac{1}{2} \left( \int_{-2}^{0} (-2 - x) \sin \frac{n\pi x}{2} \, dx + \int_{0}^{2} (2 - x) \sin \frac{n\pi x}{2} \, dx \right).
\]

\( b_n \) can be found by Euler-Fourier formula.

(c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -1? \) At \( x = 4? \)

At \( x = -1, f \) is continuous. So \( F(-1) = f(-1) = -1. \)

At \( x = 4, f \) is discontinuous. So \( F(4) = \frac{f(4)+f(4)}{2} = \frac{2+(-2)}{2} = 0. \)

(d) (4 points) Consider the even periodic extension, of period \( T = 4, \) of \( f(x). \) Sketch 3 periods, on the interval \(-6 < x < 6, \) of this even periodic extension.

(e) (3 points) Find the constant term of the Fourier series of the periodic function described in (d).

\[
a_0 = \frac{1}{2} \left( \frac{2}{L} \int_{0}^{L} f(x) \, dx \right) = \frac{1}{2} \int_{0}^{2} (2-x) \, dx = \frac{1}{2} \left( 2x - \frac{x^2}{2} \right)_{x=0}^{2} = \frac{1}{2} (4 - 2 - 0 - 0) = 1.
\]

(f) (2 points) To what value does the Fourier series of this even periodic extension converge at \( x = 4? \)

\[
F(4) = \frac{f(4)+f(4)}{2} = \frac{2+2}{2} = 2.
\]
15. (16 points) Suppose the temperature distribution function $u(x, t)$ of a rod that has both ends perfectly insulated is given by the initial-boundary value problem

$$
3u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0 \\
u_x(0, t) = 0, \quad u_x(\pi, t) = 0. \\
u(x, 0) = 9 + 16 \cos(2x) - 25 \cos(5x).
$$

(a) (12 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

Note that $a^2 = 3$ and $L = \pi$.

The general solution is

$$
U(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{4n^2 \pi^2}{\pi^2 L^2} t} \cos \frac{n \pi x}{L}.
$$

In particular (with $t = 0$),

$$
U(x, 0) = c_0 + \sum_{n=1}^{\infty} c_n \cos n \pi x.
$$

Compare this with the given initial condition $u(x, 0) = 9 + 16 \cos(2x) - 25 \cos(5x)$,

$$
c_2 = 16, \quad c_5 = -25, \quad c_0 = 9 \quad \text{and} \quad c_n = 0 \quad \text{if} \quad n \neq 0, 2, 5.
$$

Then the particular solution is

$$
U(x, t) = 9 + 16 e^{-\frac{3(2^2 \pi^2)}{\pi^2 \pi^2 L^2} t} \cos 2x - 25 e^{-\frac{3(5^2 \pi^2)}{\pi^2 \pi^2 L^2} t} \cos 5x.
$$

(b) (2 points) What is $\lim_{t \to \infty} u(\frac{\pi}{2}, t)$?

$$
9
$$

(c) (2 points) Suppose the initial condition is, instead, $u(x, 0) = 8 + 36 \cos(6x) + 49 \cos(7x)$. Will the limit $\lim_{t \to \infty} u(\frac{\pi}{2}, t)$ be higher than, lower than, or equal to the temperature you found in part (b)?

With the new initial condition, $\lim_{t \to \infty} u(\frac{\pi}{2}, t) = 8$ which is lower than what we found in part (b).
16. (20 points) Suppose the displacement $u(x, t)$ of a piece of flexible string is given by the initial-boundary value problem

$$
\begin{align*}
9u_{xx} &= u_t, & 0 < x < 2, & t > 0 \\
 u(0, t) &= 0, & u(2, t) &= 0, \\
 u(x, 0) &= 0, & u_t(x, 0) &= 4 - x^2.
\end{align*}
$$

(a) (3 points) What is the physical meaning of the boundary conditions?

Two ends of the string are clamped in fixed positions at the horizontal level so they are held motionless at all time.

(b) (3 points) What is the propagation speed of the standing waves?

$$
a = 3.
$$

(c) (2 points) When $t = 0$, what is the velocity of the vibrating string at $x = \frac{1}{2}$?

$$
\begin{align*}
u_x(\frac{1}{2}, 0) &= 4 - \left(\frac{1}{2}\right)^2 = 4 - \frac{1}{4} = \frac{15}{4}.
\end{align*}
$$

(d) (6 points) In what specific form will its general solution appear?

$$
\begin{align*}
(1) & \quad u(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{3n\pi t}{2} \sin \frac{n\pi x}{2}, & (2) & \quad u(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{3n\pi t}{2} \cos \frac{n\pi x}{2} \\
(3) & \quad u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{3n\pi t}{2} \cos \frac{n\pi x}{2}, & (4) & \quad u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{3n\pi t}{2} \sin \frac{n\pi x}{2}
\end{align*}
$$

(e) (3 points) **TRUE** or **FALSE**: The coefficients of the solution in part (d) above can be found using the integral

$$
C_n = \frac{2}{3n\pi} \int_0^2 (4 - x^2) \sin \frac{n\pi x}{2} \, dx.
$$

(f) (3 points) **TRUE** or **FALSE**: As $t \to +\infty$, the solution $u(x, t)$ in (e) will reach a limit.

The limit does not exist.