6. (14 pts) Write the following function in terms of unit step functions, and find its Laplace transform.

\[ g(t) = \begin{cases} 
  t^2 + 1 & 0 \leq t < 1 \\
  e^{-3t} + 1 & 1 \leq t < 2 \\
  1 & t \geq 2 
\end{cases} \]

\[ g(t) = (1 - u_1(t))(t^2 + 1) + (u_1(t) - u_2(t))(e^{-3t} + 1) + u_2(t) \]
\[ = t^2 + 1 + u_1(t)(e^{-3t} + 1 - t^2 - 1) + u_2(t)(1 - e^{-3t} - 1) \]
\[ = t^2 + 1 + u_1(t)(e^{-3t} - t^2) - u_2(t)e^{-3t} \]

\[ G(s) = \mathcal{L}\{g(t)\} = \frac{2}{s^3} + \frac{1}{s} + e^{-s}\mathcal{L}\{e^{-3(t+1)} - (t+1)^2\} - e^{-2s}\mathcal{L}\{e^{-3(t+2)}\} \]
\[ = \frac{2}{s^3} + \frac{1}{s} + e^{-s}\mathcal{L}\{e^{-3}e^{-3t} - t^2 - 2t - 1\} - e^{-2s}\mathcal{L}\{e^{-6}e^{-3t}\} \]
\[ = \frac{2}{s^3} + \frac{1}{s} + e^{-s}\left[\frac{e^{-3}}{s + 3} - \frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] - e^{-2s}\frac{e^{-6}}{s + 3} \]
8. (12 pts) Find the inverse Laplace transform of:

\[ F(s) = \frac{s^2 - 4}{s^3 + 6s^2 + 9s} \]

First simplify \( F(s) \) using partial fractions:

\[
F(s) = \frac{s^2 - 4}{s(s^2 + 6s + 9)} = \frac{s^2 - 4}{s(s + 3)^2} = \frac{a}{s} + \frac{b}{s + 3} + \frac{c}{(s + 3)^2} = \frac{a(s + 3)^2 + bs(s + 3) + cs}{s(s + 3)^2}.
\]

Therefore,

\[
s^2 - 4 = a(s^2 + 6s + 9) + b(s^2 + 3s) + cs = (a + b)s^2 + (6a + 3b + c)s + 9a.
\]

Solving the system

\[
\begin{align*}
1 &= a + b \\
0 &= 6a + 3b + c \\
-4 &= 9a
\end{align*}
\]

Hence,

\[
a = \frac{-4}{9}, \quad b = \frac{13}{9}, \quad c = \frac{-5}{3}.
\]

\[
F(s) = -\frac{4}{9} \frac{1}{s} + \frac{13}{9} \frac{1}{s + 3} - \frac{5}{3} \frac{1}{(s + 3)^2}.
\]

Finally,

\[
f(t) = \mathcal{L}^{-1}(F(s)) = -\frac{4}{9} + \frac{13}{9}e^{-3t} - \frac{5}{3}te^{-3t}.
\]
9. (20 pts) Solve the following initial value problem:

\[ y'' + 4y' + 8y = e^{2t} - 2\delta(t - 2\pi), \quad y(0) = 2, \quad y'(0) = 0 \]

Take the Laplace transforms of both sides and simplify:

\[
(s^2 + 4s + 8)L\{y\} - 2s - 8 = \frac{1}{s - 2} - 2e^{-2\pi s}
\]

We will find the inverse transform of the right-hand side in 2 parts. First note that the \((s^2 + 4s + 8)\) part of the denominators is an irreducible quadratic and proceed accordingly. By completing the squares, it can be rewritten as \(s^2 + 4s + 8 = (s + 2)^2 + 2^2\).

**Part I.**

By partial fractions,

\[
\frac{2s^2 + 4s - 15}{(s - 2)(s^2 + 4s + 8)} = \frac{a}{s - 2} + \frac{bs + c}{s^2 + 4s + 8}.
\]

Solve the equation above to get: \(a = \frac{1}{20}, b = \frac{39}{20}, c = \frac{154}{20}\).

Therefore,

\[
L\{y\} = \frac{1}{20} \left( \frac{1}{s - 2} + \frac{1}{s^2 + 4s + 8} \right) + \frac{1}{20} \left( \frac{39s + 154}{s^2 + 4s + 8} \right) + \frac{1}{20} \left( \frac{39(s + 2) + 76}{(s + 2)^2 + 2^2} \right)
\]

The inverse transform is

\[
y_1(t) = \frac{1}{20} e^{2t} + \frac{39}{20} e^{-2t} \cos 2t + \frac{38}{20} e^{-2t} \sin 2t.
\]

**Part II.**

\[-e^{-2\pi s} \frac{2}{(s^2 + 4s + 8)} = -e^{-2\pi s} \frac{2}{(s + 2)^2 + 2^2}\]

Its inverse transform is \(y_2(t) = -u_{2\pi}(t)f(t - 2\pi)\).

Where \(f(t) = L^{-1} \left( \frac{2}{(s + 2)^2 + 2^2} \right) = e^{-2t} \sin 2t\).

Therefore, \(y_2(t) = -u_{2\pi}(t)e^{-2(t - 2\pi)} \sin 2(t - 2\pi)\).

The very last portion can be further simplified by the identity: \(\sin 2(t - 2\pi) = \sin 2t\).

Finally, \(y(t) = y_1(t) + y_2(t) = \frac{1}{20} e^{2t} + \frac{39}{20} e^{-2t} \cos 2t + \frac{19}{10} e^{-2t} \sin 2t - u_{2\pi}(t)e^{-2(t - 2\pi)} \sin 2t\).