This exam has 11 questions for a total of 100 points. Show all your work! In order to obtain full credit for partial credit problems, all work must be shown. Points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.

Do not write in this box.

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<table>
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1. (7 points) For each problem below, determine the order of the given differential equation; also state whether the equation is linear or nonlinear. If the equation is nonlinear, please circle the term(s) that make it so.

<table>
<thead>
<tr>
<th>Differential equations</th>
<th>Order</th>
<th>Linear/Nonlinear</th>
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<tbody>
<tr>
<td>( y'' + y' y - 2ty = 0 )</td>
<td>2</td>
<td>Nonlinear</td>
</tr>
<tr>
<td>( y''' - e^{-5t}y' + (\sin t)y = 4t^2 - 7 )</td>
<td>3</td>
<td>Linear</td>
</tr>
<tr>
<td>( 2t^2 y^{(4)} + ty' - 6y = (12 - t - t^2)e^t )</td>
<td>4</td>
<td>Linear</td>
</tr>
<tr>
<td>( \frac{dy}{dt} + 3y = 6 - \tan 2t )</td>
<td>1</td>
<td>Linear</td>
</tr>
<tr>
<td>( \left( \frac{d^3 y}{dt^3} \right)^2 + \left( \frac{d^2 y}{dt^2} \right)^3 + \frac{dy}{dt} = 0 )</td>
<td>3</td>
<td>Nonlinear</td>
</tr>
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</table>

2. (6 points) Give an example of the following:

(a) (2 points) A third order partial differential equation.

(b) (2 points) A third order, linear, homogeneous, ordinary differential equation.

(c) (2 points) A first order, autonomous, ordinary differential equation.
3. (10 points) Solve explicitly the initial value problem;

\[ y' + 9xe^{3x+3x} = 0, \quad y(0) = 0. \]

We write \( y' \) by \( \frac{dy}{dx} \) as

\[
\frac{dy}{dx} + 9xe^{3x} = 0.
\]

Now separate variable \( x \) from \( y \):

\[
\frac{dy}{dx} = -9xe^{3x}.
\]

\[ e^{-y} \, dy = -9xe^{3x} \, dx. \]

We integrate both sides,

\[
\int e^{-y} \, dy = \int -9xe^{3x} \, dx.
\]

\[ -e^{-y} = -3xe^{3x} + C. \]

Now we apply an initial condition,

\[ y(0) = 0 \Rightarrow -e^{-y} = -3(0)(1) + 1 + C \Rightarrow C = -2. \]

So the particular solution is

\[ -e^{-y} = -3xe^{3x} + e^{2}. \]

Now we write it in explicit form,

\[
-\frac{1}{e^y} = -3xe^{3x} + e^{2}.
\]

\[
\frac{1}{e^y} = 3xe^{3x} - e^{2} + 2.
\]

\[
e^y = \frac{1}{3xe^{3x} - e^{2} + 2}.
\]

\[ y = \ln \left| \frac{1}{3xe^{3x} - e^{2} + 2} \right|. \]

#
4. (10 points) Find all real number $\alpha$ such that the particular solution $y = \phi(t)$ of the following initial value problem remain finite as $t \to \infty$. Justify your answer.

\[(*) \quad y' - y = \frac{1 - 2t}{e^t}, \quad y(0) = \alpha.\]

The given DE is already in a standard form with $p(t) = -2$.
So the integrating factor is

\[\mu(t) = e^{\int p(t) \, dt} = e^{\int -2 \, dt} = e^{-t}.\]

Multiply equation \((*)\) by $\mu(t)$ and grouping the LHS,

\[e^{-t}y - e^{-t}y' = \left(\frac{1 - 2t}{e^t}\right)e^{-t}\]

\[\left(e^{-t}y\right)' = \left(1 - 2t\right)e^{-t}\]

\[\frac{d}{dt} \left(e^{-t}y\right) = \left(1 - 2t\right)e^{-t}\]

Integrate both sides:

\[\int \left(e^{-t}y\right)' \, dt = \int \left(1 - 2t\right)e^{-t} \, dt\]

\[e^{-t}y = \frac{1 - 2t}{2}e^{-t} + \frac{-2t}{e^t} + C\]

\[y = \frac{2t - 1}{2}e + \frac{-2t}{e^t} + Ce^t = te^t + ce^t\]

Apply an initial condition $y(0) = \alpha$,

\[\alpha = 0 + C \Rightarrow C = \alpha\]

So the particular solution is

\[y = te^t + \alpha e^t\]

Since $\lim_{t \to \infty} e^{-t} = 0$ and $\lim_{t \to \infty} e^t = \infty$, $\alpha$ must be zero in order to make the $\lim_{t \to \infty} y(t) < \infty$. 

(If $\alpha > 0$, then $\lim_{t \to \infty} y(t) = \frac{\infty}{-\alpha}$ if $\alpha > 0$, 

(If $\alpha < 0$, then $\lim_{t \to \infty} y(t) = \frac{-\infty}{\alpha}$ if $\alpha < 0$. 

\[\#\]
5. (10 points) Solve the initial value problem:

\[(2x \sin y - y \cos x) + (x^2 \cos y - \sin x) y' = 0, \quad y\left(\frac{\pi}{2}\right) = 0.\]

You may leave your answer in implicit form.

We have

\[M(x,y) = 2x \sin y - y \cos x, \quad N(x,y) = x^2 \cos y - \sin x.\]

Then

\[M_y = (2x \sin y - y \cos x)_y = 2x \cos y - \cos x, \quad N_x = (x^2 \cos y - \sin x)_x = 2x \cos y - \cos x,\]

Since \(M_y = N_x\), this equation is exact.

Consider

\[\psi = M = \int M \, dx = \int (2x \sin y - y \cos x) \, dx = x^2 \sin y - y \sin x + c_1(y). \tag{1}\]

\[\psi = N = \int N \, dy = \int (x^2 \cos y - \sin x) \, dy = x^2 \sin y - y \sin x + c_2(x). \tag{2}\]

By comparing (1) and (2), \(c_1(y) = c_2(x) = 0\). So

\[\psi(x,y) = x^2 \sin y - y \sin x.\]

Therefore the general solution is

\[x^2 \sin y - y \sin x = C.\]

Now we impose an initial condition,

\[y\left(\frac{\pi}{2}\right) = 0 \implies \left(\frac{\pi}{2}\right)^2 \sin 0 - 0 \sin \frac{\pi}{2} = C \implies C = 0\]

So the particular solution is

\[x^2 \sin y - y \sin x = 0.\]
6. (6 points) Find the largest interval on which the solution of

\[(2 - \ln t)y' + 3y = 4t^2, \quad y(2) = 10\]

is guaranteed to exist, without solving the IVP itself. (Hint: \( e \) is approximately 2.718)

Rewrite the DE in standard form

\[
y' + \frac{3}{2-\ln t} y = \frac{4t^2}{2-\ln t}. \quad (t_0 = 2)
\]

First \( \ln t \) is defined only if \( t > 0 \).

Also the discontinuity happens when \( 2-\ln t = 0 \iff \ln t = 2 \iff t = e^2 \).

So the interval of validity is \((0, e^2)\). #

7. (6 points) A 600-gallon tank is initially filled with 400 gallons of water with a salt concentration of 1 pound per gallon. A salt water mixture with a concentration of 3 pounds per gallon enters the tank at the rate of 20 gallons per minute. Then a thoroughly mixed solution leaves the tank at the rate of 24 gallons per minute. Let \( Q(t) \) be the quantity of salt in the tank at time \( t \). (pounds) Find the initial value problem which accurately describes the situation.

We have

\[V_0 = 400 \text{ gallons}, \quad Q(0) = (400 \text{ gal})(1 \text{ lb/gal}) = 400 \text{ pounds}\]

\[r_i = 20 \text{ gal/min}, \quad c_i = 3 \text{ lb/gal}, \quad r_o = 24 \text{ gal/min}\]

So the IVP modeling this problem is

\[Q'(t) = (20)(3) - 24 \frac{Q(t)}{400 + (20-24)t}, \quad Q(0) = 400\]

\[= 60 - \frac{24 Q(t)}{400 - 4t}, \quad Q(0) = 400\]

\[= 60 - \frac{6 Q(t)}{100 - t}, \quad Q(0) = 400\]

#
8. (13 points) Consider the following differential equation

$$y' = y(1 + y)^2 (2 - y)^3.$$ 

(a) (3 points) Find all of its equilibrium solutions.

Set $y' = 0$; $y(1 + y)^2 (2 - y)^3 = 0$

$$y = 0, -1, 2$$ are all equilibrium solutions.

(b) (4 points) Classify the stability of each equilibrium solution. Justify your answer.

$$\begin{align*}
y & \downarrow \\
y = 2 & \text{ is stable.} \\
y = 0 & \text{ is unstable} \\
y = -1 & \text{ is semistable.}
\end{align*}$$

Phase Line

(c) (2 points) If $y(-3) = -1$, what is $y(-9)$? Without solving the equation, briefly explain your conclusion.

Since $y_0 = -1$ which is an equilibrium solution, $y(-9) = -1$.
(Indeed, $y(t) = -1$ for all $t$.)

(d) (4 points) Suppose $y(3) = \alpha$ and $\lim_{{t \to \infty}} y(t) = 2$. Find the value(s) of $\alpha$.

From phase line, $\lim_{{t \to \infty}} y(t) = 2$ happens when $\alpha > 0$ or $(0, \infty)$.  

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9. (10 points) Find the particular solutions of the following initial value problem. Express your answer in terms of real valued functions.

\[ y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = -5. \]

Find also \( \lim_{t \to \infty} y(t) \).

Char eq: \( r^2 + 2r + 5 = 0 \).

\[ r = -1 \pm \frac{\sqrt{-4 - 20}}{2} = -1 \pm 2i. \]  \( \lambda = -1, \mu = 2 \).

So the general solution is \( y = e^{-t} (c_1 \cos 2t + c_2 \sin 2t) \).

We compute \( y' = e^{-t} (-2c_1 \sin 2t + 2c_2 \cos 2t) - e^{-t} (c_1 \cos 2t + c_2 \sin 2t) \).

Now let's apply initial conditions:

\[
\begin{align*}
  y(0) &= 1 \implies 1 = c_1, \\
  y'(0) &= -5 \implies -5 = 2c_2 - c_1.
\end{align*}
\]

So the particular solution is

\[ y = e^{-t} (\cos 2t - 2 \sin 2t). \]

Therefore

\[ \lim_{t \to \infty} y(t) = 0. \]
10. (12 points) Given that \( y_1(t) = t^{-1} \) is a solution to the equation,
\[ t^2y'' + 3ty' + y = 0, \quad t > 0. \]

Use the method of reduction of order to find another solution \( y_2 \).

Let \( y_2(t) = v \cdot t^{-1} \) be the second solution of the DE above.

Compute \( y_2' = -v \cdot t^{-2} + v' \cdot t^{-1} \)
\[ y_2'' = 2v \cdot t^{-3} - 2t^{-2}v' \cdot t^{-2} + v'' \cdot t^{-1} = 2v \cdot t^{-3} - 2v' \cdot t^{-2} + v'' \cdot t^{-1}. \]

Since \( y_2 \) is a solution, we have
\[ 0 = t^2y_2'' + 3ty_2' + y_2 = t^2(2v \cdot t^{-3} - 2v' \cdot t^{-2} + v'' \cdot t^{-1}) + 3t(-v \cdot t^{-2} + v' t^{-1}) + v t^{-1} \]
\[ = \frac{2v t^{-1} - 2v'}{t^{-1}} + \frac{3v' t^{-1} + 3v}{t^{-1}} \]
\[ = \frac{v' + v'' t}{t}. \]

Now write \( u = v' \); \( 0 = u + u' t \).
\[ \frac{du}{dt} = -u \]
\[ \int \frac{du}{u} = -\int t \ dt \]
\[ \ln |u| = -\ln |t| + c \]
\[ u = e^{-\ln t + c'} = e^{\ln t^{-1} c'} = c t^{-1} \quad \text{where} \quad c = c'. \]

But \( u = v' \)
\[ v' = ct^{-1} \]
\[ \int dv = ct^{-1} \ dt \]
\[ v = c \ln t + k \quad \text{(Now choose} \quad c = 1; k = 0 \).

So the second solution is
\[ y_2(t) = v \cdot t^{-1} = (c \ln t + k) t^{-1} = ct^{-1} \ln t + kt^{-1} \]
\[ \text{or simply} \quad y_2(t) = t^{-1} \ln t. \]
11. (10 points) Consider the second order linear differential equation
\[ t^2y'' + 2ty' + 3y = 0. \]

Suppose \( y_1(t) \) and \( y_2(t) \) are two fundamental sets of solutions of the equation satisfying
\[ y_1(1) = 4, \quad y_1'(1) = 8, \quad y_2(1) = 1, \quad y_2'(1) = 3. \]

Compute their Wronskian \( W(y_1, y_2)(t) \) as a function of \( t \).

Rewrite the DE into the standard form,
\[ y'' + \frac{2}{t}y' + \frac{3}{t^2}y = 0. \]

Notice that \( p(t) = \frac{2}{t} \). So by Abel's theorem,
\[ W(y_1, y_2)(t) = Ce^{-\int\frac{2}{t}dt} = Ce^{-2\ln t} = Ce^t. \] \(
\)

By definition,
\[ W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t). \]

With \( t = 1 \), we have
\[ W(y_1, y_2)(1) = y_1(1)y_2'(1) - y_1'(1)y_2(1) = 4(3) - 8(1) = 4. \] \( \text{(1)} \)

Also by (\( \ast \)), \( W(y_1, y_2)(t) = c(t)^{-2} = c \).

Combine (1) and (2), \( c = 4. \)

Therefore
\[ W(y_1, y_2)(t) = 4t^{-2} \text{ by (\( \ast \)).} \]