This exam contains 10 questions on 9 pages (including this title page). This exam is worth a total of 100 points. The exam is broken into two parts. There are six multiple choice questions, each worth 5 points, and 4 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

**No notes, books or calculators may be used during the exam.**

Please, Box Your Final Answer (when possible).

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Multiple Choice Section

1. (8 points) Match the differential equation with the appropriate direction field.

(A)  

(B)  

(C)  

(D)  

(a) \( y'(x) = y(x) + x \)  
(b) \( y'(x) = y(x) - x \)  
(c) \( y'(x) = y(x)(1 - y(x)) \)  
(d) \( y'(x) = x(1 - x) \)  

Points  

Points  

Points  

Points  

\( y' = 0 , \quad y(x) = -x \)  
\( y' = 0 , \quad y(x) = x \)  
\( y' = 0 , \quad y = 0 , \quad y = 1 \)  
\( y' = 0 , \quad x = 0 , \quad x = 1 \)
2. (8 points) Match the differential equation with the solution.

\begin{align*}
(A) \quad & y'(x) = -2y(x) \quad \text{and} \quad y(x) = C_1 \cos(2x) + C_2 \sin(2x) \\
(B) \quad & y'(x) = \frac{3}{x} y(x) \quad \text{and} \quad y(x) = \sin(x) + C \\
(C) \quad & y'(x) = \cos(x) \quad \text{and} \quad y(x) = C e^{-2x} \\
(D) \quad & y''(x) = 4y(x) \quad \text{and} \quad y(x) = x^3 + C
\end{align*}

3. (8 points) Classify the following differential equations in terms of

(i) order
(ii) linear/non-linear
(iii) ODE/PDE

\begin{tabular}{cccc}
 Order & L/n.L & ODE/PDE \\
\hline
(a) \quad & u_{xx}(x, y) + u_{yy}(x, y) = x^2 + y^2 & 2nd \ & linear \ & PDE \\
(b) \quad & y'(x) + y^2(x) = \sin(x) & 1st \ & non-linear \ & ODE \\
(c) \quad & y'(x) - xy(x) = e^x & 1st \ & linear \ & ODE \\
(d) \quad & x^2y'(x) + \sin(x)y(x) = 1 & 1st \ & linear \ & ODE
\end{tabular}
Partial Credit Section

4. (8 points) Solve the following initial value problem:

\[ xy' = y^2, \quad y(1) = 2. \]

Separable form

\[ \frac{y'}{y^2} = \frac{1}{x} \]

Integrate

\[ \int \frac{y'}{y^2} \, dx = \int \frac{1}{x} \, dx \]

\[ -\frac{1}{y} = \ln |x| + C \]

From I.C.

\[ -\frac{1}{2} = 0 + C \Rightarrow C = -\frac{1}{2} \]

Solution

\[ y = \frac{-1}{\ln |x| + C} = \frac{-1}{\ln |x| - \frac{1}{2}} \]
5. (10 points) Solve the following initial value problem:

\[ xy' = y + x, \quad y(0) = 1. \]

**Standard form for linear eqn:**

\[ y' + \frac{-1}{x} y = 1 \]

\[ \mu y' + \frac{-1}{x} \mu y = \mu \]

Find \( \mu \) s.t.

\[ \mu' = \frac{-1}{x} \mu \implies \mu = \frac{1}{x} \]

\[ (\mu y)' = \mu \]

\[ \mu y = \int \mu \, dx + C = \int \frac{1}{x} \, dx + C = \ln |x| + C \]

\[ y = x(\ln |x| + C) \]
6. A tank initially contains 100 liters of fresh water. A mixture containing 10 grams of salt per liter is poured into the tank at the rate of 2 liters per minute. The well-stirred mixture is allowed to leave the tank at the same rate.

(a) (2 points) Introduce the variables and their meaning.

\[ W(t) \] - amount of water in the tank
\[ S(t) \] - amount of salt in the tank
\[ \text{In} = 2 \text{ Lit/min.} \quad \text{Out} = \text{In} \]
\[ C_{\text{In}} = 10 \frac{g}{\text{Lit}} \quad C_{\text{Out}} = \frac{S(t)}{W(t)} \]

(b) (4 points) Write the differential equations, and give the initial conditions, that describe this event.

\[
\begin{align*}
W'(t) &= \text{In} - \text{Out} = 2 - 2 = 0, \quad W(0) = 100 \\
S'(t) &= \text{In} C_{\text{In}} - \text{Out} C_{\text{Out}} = 2 \cdot 10 - 2 \frac{S(t)}{W(t)} \\
S(0) &= 0
\end{align*}
\]

(c) (8 points) Solve the initial value problem.

\[
W(t) = 100 \Rightarrow \\
S'(t) = 20 - \frac{2}{100} S(t) \Rightarrow S' + \frac{1}{50} S = 20 \quad \Rightarrow \quad m' = \frac{1}{50} m \Rightarrow m(t) = e^{\frac{t}{50}}
\]

\[
e^{\frac{t}{50}} S = \int (e^{\frac{t}{50}} S)' dt = \int 20 e^{\frac{t}{50}} dt = 20 \cdot 50 e^{\frac{t}{50}} + C
\]

\[ S = 1000 + C e^{\frac{t}{50}} \]

From I.C. \( S(0) = 0 \Rightarrow C = -1000 \]

\[ S(t) = 1000 \left( 1 - e^{-\frac{t}{50}} \right) \]
(d) (4 points) Find the time $T$ when the concentration of salt in the tank is 5 grams per liter.

Concentration of salt in the tank

$$ S(t) = \frac{1000 \left(1 - e^{\frac{-t}{50}}\right)}{100} = 5 \frac{\text{gr}}{\text{L}}. $$

$$ 1 - e^{\frac{-t}{50}} = 0.5 \Rightarrow e^{\frac{-t}{50}} = 0.5 \Rightarrow \frac{-t}{50} = \ln\left(\frac{1}{2}\right) $$

$$ t = 50 \ln(2) = 50 \ln 2 $$

(e) (2 points) What is the limit concentration of salt in the tank?

$$ \lim_{t \to +\infty} \frac{1000 \left(1 - e^{\frac{-t}{50}}\right)}{100} = 10 = C_{\text{in}}. $$
7. (8 points) Without solving the differential equation, find the largest interval where the initial value problem is guaranteed to have a unique solution.

\[(x^2 - 1)y'(x) + e^{x^2}y(x) = x, \quad y(0) = 3.1415.\]

Standard form

\[y' + \frac{e^{x^2}}{x^2 - 1} y = \frac{x}{x^2 - 1}\]

Problem points \(x = 1, \ x = -1\)

Largest interval \([-1, 1]\).

8. (8 points) For which values \(y_0\) of the initial condition the following initial value problem is guaranteed to have a unique solution (locally)?

\[y'(x) = \sqrt{y(1 + x^2)}, \quad y(0) = y_0.\]

\[y'(x) = f(x, y) = \sqrt{y} (1 + x^2)\]

- \(f(x, y) = \sqrt{y} (1 + x^2)\) is continuous everywhere \(y \geq 0\).

- \(f_y(x, y) = \frac{2}{\sqrt{y}} \left[ \sqrt{y} (1 + x^2) \right] = \frac{1}{2} \frac{1}{\sqrt{y}} (1 + x^2)\) is continuous for \(y > 0\).

For \(y_0 > 0\) there is a unique solution (locally).
9. Consider the following autonomous differential equation

\[ y' = (y^2 - 9)(y + 3)y. \]  

(a) (4 points) Sketch the function \( f(y) = (y^2 - 9)(y + 3)y = (y-3)(y+3)(y+3)y \)

(b) (4 points) Identify the equilibrium solutions of (1) and classify their stability.

\[ y = 3 \quad \text{unstable} \]
\[ y = 0 \quad \text{stable} \]
\[ y = -3 \quad \text{semistable} \]

(c) (2 points) Find the limit value of the solution that satisfies the initial condition \( y(-4) = 4 \).

\[ \lim_{t \to +\infty} y(t) = +\infty \]
10.
(a) (4 points) Check whether the following differential equation is exact or not. Do not solve!

\[
(2xy^3 - \sin(x)) + (3x^3y + \cos(x))y' = 0.
\]

\[
\frac{\partial}{\partial y} (2xy^3 - \sin(x)) = 6xy^2
\]

\[
\frac{\partial}{\partial x} (3x^3y + \cos(x)) = 6xy^3 - \sin(x)
\]

Not exact

(b) (8 points) Solve the following exact differential equation:

\[
\frac{(xy^2 - y \sin(xy) + x)}{R_x} + \frac{(x^2y - x \sin(xy) + y)}{R_y} y' = 0.
\]

\[
R_x = xy^2 - y \sin(xy) + x
\]

\[
R(y) = \frac{1}{2}x^2y^2 + \cos(xy) + \frac{1}{2}x^2 + \frac{1}{2}y^2 + g(x)
\]

\[
R_y = x^2y - x \sin(xy) + y
\]

\[
R(x,y) = \frac{1}{2}x^2y^2 + \cos(xy) + \frac{1}{2}x^2 + \frac{1}{2}y^2 + g(x)
\]

Solution (implicit)

\[
\frac{1}{2}x^2y^2 + \cos(x,y) + \frac{1}{2}x^2 + \frac{1}{2}y^2 = C
\]