This exam has 11 questions for a total of 150 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached on the last page of the exam.

**YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.**
1. (12 points) Solve the initial value problem

\[ 2y' + y = 3t, \quad y(0) = 3. \]
2. (15 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, \( x' = Ax \), where \( A \) is a \( 2 \times 2 \) matrix of real numbers. Based solely on the information given in each part, determine the type and stability of the system’s critical point at \((0,0)\). Justify your answer.

(a) (3 points) The coefficient matrix is \[
\begin{bmatrix}
1 & 0 \\
2 & 3
\end{bmatrix}.
\]

(b) (3 points) One of the eigenvalues of \( A \) is \( r = -3\sqrt{3}i \).

(c) (3 points) The characteristic equation of \( A \) can be written as \( r^2 + 6r + 6 = 0 \).

(d) (3 points) The general solution is \( x(t) = C_1 e^{5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} t \\ -2t + 1 \end{bmatrix} \).

(e) (3 points) The general solution is \( x(t) = C_1 e^{-2t} \begin{bmatrix} \cos 8t - 8 \sin 8t \\ 5 \cos 8t \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} \sin 8t + 8 \cos 8t \\ 5 \sin 8t \end{bmatrix} \).
3. (14 points) Consider the nonlinear system:

\[ x' = x + x^2 + y^2 \]
\[ y' = y - xy \]

(a) (6 points) Find all (real) critical points of the system.

(b) (8 points) Linearize the system about each critical points. Then classify the type and stability of each critical points by examining the linearized system. Be sure to clearly state the linearized system’s matrix and its eigenvalues.
4. (14 points) Consider the partial differential equation with its boundary conditions,

\[ u_t = 3u_{xx} - u, \quad u(0, t) = 0, \quad u_x(\pi, t) = 0. \]

(a) (6 points) Use separation of variables to change the partial differential equation into two ordinary differential equations. Namely let \( u(x, t) = X(x)T(t) \) and find the ordinary differential equations \( X(x) \) and \( T(t) \) satisfy. Use \( -\lambda \) for the separation constant.

(b) (4 points) What new boundary conditions must the equation of \( X(x) \) satisfy after using the technique of separation of variables?

(c) (4 points) State TRUE/FALSE with reason. If new boundary conditions are

\[ u(0, t) = 2, \quad u_x(\pi, t) = 0, \]

then it can be rewritten as \( X(0) = 2 \) and \( X'(\pi) = 0 \) after using the technique of separation of variables.
5. (15 points) Let the function $f(x)$ be given on $0 \leq x \leq 2$ by:

$$f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ -x + 1, & 1 \leq x \leq 2 \end{cases}.$$

(a) (4 points) Consider the **odd** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $[-6, 6]$, of this function.

(b) (4 points) To what value does the Fourier series of this odd periodic extension converge at $x = -3$? At $x = 0$? At $x = \frac{3}{2}$?

(c) (4 points) Consider the **even** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $[-6, 6]$, of this function.

(d) (3 points) State TRUE/FALSE with reason. For the same even periodic extension mentioned in part (c), the Fourier cosine coefficients are given by

$$a_n = 2 \int_0^1 \cos \left( \frac{n\pi x}{2} \right) \, dx + \int_1^2 (1 - x) \cos \left( \frac{n\pi x}{2} \right) \, dx.$$
6. (12 points) Find all eigenvalues and corresponding eigenfunctions of the two-point boundary value problem

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(\pi) = 0.$$ 

Make sure to consider all three cases: $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$. 
7. (10 points) Find the Fourier series for the function given by

\[ f(x) = \begin{cases} 
0, & -\pi \leq x < 0 \\
x^2, & 0 \leq x < \pi 
\end{cases}, \quad f(x + 2\pi) = f(x). \]
8. (12 points) Suppose you have a thin wire of length 2 cm, with the thermal diffusivity 0.01 cm$^2$/s and an initial temperature distribution of $3 \sin \pi x - 5 \sin 3\pi x + 7 \sin 5\pi x$ for $0 < x < 2$. Suppose that both ends are embedded in ice (temperature 0°C). Find the temperature distribution function.
9. (14 points) Suppose a thin rod 10 cm long is insulated along its sides and made of a copper (its thermal diffusivity is $1.14 \text{ cm}^2/\text{s}$.) Also suppose that the left end is held at constant temperature of $40^\circ\text{C}$ and the right end is held at constant temperature of $60^\circ\text{C}$. Assume that the initial temperature distribution is $3x + 40$, where $x$ is a position inside the rod from the left end.

(a) (3 points) What is the temperature at the middle of the rod at the beginning of the experiment ($t = 0$)?

(b) (8 points) Find the temperature $u(x, t)$ of the rod at any time $t$ and at any point $x$ inside the rod.

(c) (3 points) Find the steady state solution at the middle of the rod.
10. (12 points) Consider an elastic string of length 10 cm whose ends are held fixed. The string set in motion from its equilibrium position with an initial velocity \( g(x) \) where

\[
g(x) = \begin{cases} 
1, & \text{if } 2 < x < 5 \\
0, & \text{otherwise}
\end{cases}.
\]

(a) (10 points) Find the displacement \( u(x, t) \) which satisfies the wave equation \( 4u_{xx} = u_{tt} \).

(b) (2 points) State TRUE or FALSE with reason. The string will eventually come to rest and the displacement will be zero.
11. (20 points) True or False:

(a) (4 points) The Fourier series of \( f(x) = 2 + \sin x - 4 \cos x \) is itself.

(b) (4 points) Every separable equation is exact.

(c) (4 points) Given that \( y = t - 2 \sin t \) is a solution of \( y'' + 2y' = g(t) \). Then \( y_1 = 3t - 6 \sin t \) is also a solution of the same equation.

(d) (4 points) A mass-spring system described by the equation \( 2y'' + 8y = 3 \cos(2t) \) exhibits resonance.

(e) (4 points) The Fourier series of 
\[
    f(x) = 2x^5 + 3 \sin 4x, \quad -\pi \leq x < \pi, \quad f(x + 2\pi) = f(x)
\]
is a Fourier sine series.
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<td>$t^n; \quad n = \text{positive integer}$</td>
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<td>$\frac{n!}{s^{n+1}}, \quad s &gt; 0$</td>
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