MATH 251
Final Examination
August 14, 2014
FORM A

Name:__________________
Student Number:__________________
Section:__________________

This exam has 11 questions for a total of 150 points. Show all your work! In order to obtain full credit for each problems, all work must be shown. Credit will not be given for an answer not supported by work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.

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1. (20 points) True or false. Don’t forget to justify your answer.

(a) (4 points) $e^{-t} \cos(3t)y' = y$ is a first order, linear, separable ordinary differential equation.

(b) (4 points) $(x + 2)xe^x \sin y + (x^2e^x \cos y)y' = 0$ is NOT an exact equation.

(c) (4 points) If $y_1$ is a solution of $y'' + 4y' + 5y = 0$, so is $3y_1$.

(d) (4 points) $\sin(x^2)$ can be represented by a Fourier sine series.

(e) (4 points) Every non-zero constant function is an even function but is NOT an odd function.
2. (12 points) Solve the initial value problem

\[ y' + 2y = t, \quad y(0) = \frac{3}{4}. \]

You may use either the method of integrating factor or Laplace transform to solve this problem.
3. (8 points) Let \( y(t) \) be the particular solution of the following initial value problem

\[
y' = (-y)(2 - y)(y + 4), \quad y(3) = -2.
\]

What is \( \lim_{t \to \infty} y(t) \)? (Hint: phase line)

4. (12 points) Find a second order linear equation which has

\[
y(t) = c_1 e^t + c_2 e^{-2t} + 3t - 1
\]

as its general solution.
5. (12 points) Solve the following linear system,

\[ \begin{align*}
x_1' &= 3x_1 + 9x_2, \quad x_1(0) = 2, \\
x_2' &= -x_1 - 3x_2, \quad x_2(0) = 4.
\end{align*} \]

Hint: Rewrite the problem to \( x' = Ax \) where \( A \) is a \( 2 \times 2 \) matrix.
6. (14 points) Consider the 2-point boundary value problem:

\[ X'' + \frac{\lambda \pi^2}{4} X = 0, \quad X'(0) = 0, \quad X'(2) = 0. \]

Find ALL eigenvalues and the corresponding eigenfunctions. Check whether there are any eigenvalues for \( \lambda > 0 \), for \( \lambda = 0 \), for \( \lambda < 0 \). Show your work!
7. (14 points) Consider the partial differential equation with its boundary conditions,

\[ U_{xt} - U_{xx} - U_{t} = 0, \]
\[ U(0, t) = 0, \quad U_x(5, t) = 0. \]

(a) (8 points) Use the technique of separation of variables to change the above PDE into two ordinary differential equations. Namely, let \( U(x, t) = X(x)T(t) \) and find the ordinary differential equations \( X(x) \) and \( T(t) \) satisfy.

(b) (3 points) What new boundary conditions must the equation of \( X(x) \) satisfy after using the technique of separation of variables?

(c) (3 points) True or false. If new boundary conditions are \( U(0, t) = 1 \) and \( U_x(5, t) = 1 \), then it can be rewritten as \( X(0) = 1 \) and \( X'(5) = 1 \) after using the technique of separation of variables.
8. (14 points) Suppose a thin rod 10 cm long is insulated along its sides and made of a silver (its thermal diffusivity is $1.71 \text{ cm}^2/\text{sec}$). Also suppose that the left end is held at constant temperature of 30°C and the right end is held at constant temperature of 80°C. Assume that the initial temperature distribution is $6x$, where $x$ is a position inside the rod from the left end.

(a) (3 points) What is the temperature at $x = 2$ at the beginning of the experiment ($t = 0$)?

(b) (8 points) Find the temperature $U(x, t)$ of the rod at any time $t$ and at any point $x$ inside the rod.

(c) (3 points) Find the steady state solution at $x = 2$. 

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9. (14 points) Suppose a thin homogeneous rod 5 cm long is insulated along its sides and made of an aluminum (its thermal diffusivity is \(0.86 \text{ cm}^2/\text{sec.}\)) Also assume that both ends of this rod are insulated.

(a) (7 points) If the initial temperature distribution of the rod is \(x^2 + 5\) then find the eventual temperature.

(b) (7 points) If the initial temperature distribution of the rod is \(337 + 338 \cos(\pi x)\), then what is the temperature of the rod at \(x\)-units from the left end at time \(t > 0\)?
10. (14 points) The displacement $U(x,t)$ of a string of length 8 cm with both ends clamped satisfies the differential equation

$$64U_{xx} = U_{tt}.$$ 

(a) (7 points) If the initial displacement of the string is $5 \sin 3\pi x$ and the initial velocity of the string is 0. Find the displacement $U(x,t)$ of the string at any position $x$ at anytime $t > 0$.

(b) (7 points) Now assume that the initial displacement of the string in Part (a) is 0 and the initial velocity is given by $\sin \pi x$, then what is the displacement of the string at any position $x$ at anytime $t > 0$?
11. (16 points) Answer the following questions.

(a) (4 points) Let \( f(x) = x - 1, \ -3 \leq x < 3, \ f(x + 6) = f(x) \). To what value does its Fourier series converge at \( x = 20 \)? At \( x = 21 \)?

(b) (4 points) Let \( f(x) = x^2, \ 0 \leq x \leq 2 \). Sketch its even periodic extension, of period \( 2L = 4 \), of \( f(x) \) on the interval \([-8, 8]\).

(c) (4 points) Find \( \mathcal{L}\{te^{-t}\sin 2t\} \).

(d) (4 points) Classify the type and stability of the corresponding linear system near critical point \((-1, 0)\) of the nonlinear system \( x' = x + x^2 + y^2, \ y' = y - xy \).