This exam has 14 questions for a total of 150 points. Show all your work! In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.
1. (6 points) Which of the following equations is a second order nonlinear ODE?
   
   (a) \((y')^2 + y \sin t = 1\)
   
   (b) \(2t + 3 = yy''\)
   
   (c) \(y''' + 2y'' - y' = y + 1\)
   
   (d) \(y'' + e^t y' + t^2 y = t^{-\frac{1}{2}}\)

2. (6 points) Which of the following pairs of functions are linearly independent?
   
   (a) \(4t, 4^{t+5}\)
   
   (b) \(-6 \cos(2t), 2 \cos^2 t - 2 \sin^2 t\)
   
   (c) \(t^2 + 2t - 1, 3t^2 - 8t + 4\)
   
   (d) \(\sin(t + 4\pi), -3 \sin t\)

3. (6 points) The solution of the first order differentiable equation \(y' = (x - 2)(y + 4)\) is
   
   (a) \(y = Cx^2 - 2x\)
   
   (b) \(0.5y^2 + 4y = C e^{\frac{x^2}{2} - 2x} - 4\)
   
   (c) \(y = C e^{\frac{x^2}{2} - 2x} - 4\)
   
   (d) \(y = e^{\frac{x^2}{2} - 2x} - 4C\)
4. (6 points) Consider the problems below.

\[(I)\] \[y'' + \alpha y' + \beta y = 0, \quad y(0) = \gamma, \quad y'(0) = \delta\]

\[(II)\] \[y'' + \alpha y' + \beta y = 0, \quad y(0) = \gamma, \quad y(2\pi) = \delta.\]

(a) Only (I) has a unique solution for every real numbers \(\alpha, \beta, \gamma, \) and \(\delta.\)
(b) Only (II) has a unique solution for every real numbers \(\alpha, \beta, \gamma, \) and \(\delta.\)
(c) Both (I) and (II) have a unique solution for every real numbers \(\alpha, \beta, \gamma, \) and \(\delta.\)
(d) Neither is guaranteed to have a unique solution for every real numbers \(\alpha, \beta, \gamma, \) and \(\delta.\)

5. (6 points) Which of the following statements about critical point \((1,1)\) of the given nonlinear system is true?

\[\frac{dx}{dt} = x - xy\]
\[\frac{dy}{dt} = y^2 - x^2.\]

(a) \((1,1)\) is an unstable saddle point
(b) \((1,1)\) is an asymptotically stable node
(c) \((1,1)\) is a spiral point
(d) \((1,1)\) is a center

6. (6 points) Find the Fourier sine coefficient corresponding to \(n = 4, b_4,\) of the Fourier series of the periodic function,

\[f(x) = -x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad f(x + \pi) = f(x).\]

(a) \(b_4 = \frac{1}{4}\)
(b) \(b_4 = 0\)
(c) \(b_4 = -1\)
(d) \(b_4 = \frac{1}{2}\)
7. (15 points) Consider systems of linear differentiable equations of the form \( x' = Ax \), where \( A \) is a \( 2 \times 2 \) real matrix. For each pair of eigenvalues of the matrix \( A \) listed below, state the type and stability of the critical point at \((0,0)\)?

(a) (3 points) \( r_1 = 1, \ r_2 = 5 \)

(b) (3 points) \( r_1 = -2 + \sqrt{7}i, \ r_2 = -2 - \sqrt{7}i \)

(c) (3 points) \( r_1 = 6, \ r_2 = 6 \) with two linearly independent eigenvectors

(d) (3 points) \( r_1 = -4, \ r_2 = 8 \)

(e) (3 points) \( r_1 = 3i, \ r_2 = -3i \)
8. (15 points) Determine a suitable form for a particular solution $Y_p(t)$ if the method of undetermined coefficients is to be used for each of the following nonhomogeneous equation. **DO NOT** determine the values of the coefficients.

(a) (5 points) $y'' - 6y' + 9y = 3e^t + e^{3t}$

(b) (5 points) $y'' + 25y = 8 \sin 5t + e^{-2t} \cos 6t$

(c) (5 points) $y'' + 5y' - 6y = e^t + (t^2 + 7)$
9. (14 points)
   
   (a) (7 points) Find the Laplace transform of \( f(t) = te^t \cos 3t \).

   (b) (7 points) Find the inverse Laplace transform of \( F(s) = \frac{e^{-3s}}{s^2 + 25} + \frac{3s + 1}{s^2 - 2s - 15} \).
10. (18 points) TRUE or FALSE.
   
   (a) (3 points) The autonomous equation \( y' = y^3 + y^2 - 6y \) has two asymptotically stable equilibrium solutions.

   (b) (3 points) The equation \( x^2 \sin x + ye^x + (e^x + \ln y + x)y' = 0 \) is exact.

   (c) (3 points) \( \mathcal{L}\{(t + 1)(t - 1)\} = \frac{1}{s^4} - \frac{1}{s^2} \).

   (d) (3 points) A function \( f(x) = x^4 + \cos 2x \) is represented by a Fourier cosine series.

   (e) (3 points) Zero function is both even and odd.

   (f) (3 points) It is possible to separate the partial differential equation \( xu_{xx} + u_{xt} + tu_{tt} = 0 \) into two ordinary differential equations.
11. (16 points) Find all nonnegative eigenvalues and corresponding eigenfunctions of the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'\left(\frac{\pi}{4}\right) = 0. \]

Make sure to consider both cases: \( \lambda = 0 \) and \( \lambda > 0 \).
12. (18 points) Let \( f(x) = 1 - x^2 \), \( 0 < x < 1 \).

(a) (4 points) Consider the odd periodic extension, of period \( 2L = 2 \), of \( f(x) \). Sketch 3 periods, on the interval \(-3 < x < 3\).

(b) (3 points) Find the constant term, \( \frac{a_0}{2} \), of the Fourier series of the periodic function described in (a).

(c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = 0.5 \)? At \( x = 6 \)?

(d) (4 points) Consider the even periodic extension, of period \( 2L = 2 \), of \( f(x) \). Sketch 3 periods, on the interval \(-3 < x < 3\).

(e) (3 points) Write the formulas for the Fourier series coefficients of the periodic function described in (d). DO NOT EVALUATE THE INTEGRALS.
13. (18 points) Suppose a temperature distribution \( u(x, t) \) of a rod is given by the initial-boundary value problem:

\[
\begin{align*}
5u_{xx} &= u_t, \quad 0 < x < 10, \quad t > 0 \\
u(0, t) &= 30, \quad u(10, t) = 70 \\
u(x, 0) &= 4x + 30 - 6 \sin \frac{\pi x}{2} + 7 \sin 3\pi x.
\end{align*}
\]

(a) (2 points) What is the physical meaning of its boundary conditions?

(b) (4 points) Find its steady-state solution \( v(x) \).

(c) (10 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(d) (2 points) What is \( \lim_{t \to \infty} u(5, t) \)?
14. (16 points) Suppose the displacement $u(x, t)$ of a piece of flexible string that has both ends firmly fixed in places is given by the initial-boundary value problem

$9 u_{xx} = u_{tt}, \quad 0 < x < 6, \quad t > 0$

$u(0, t) = 0, \quad u(6, t) = 0$

$u(x, 0) = 0,$

$u_t(x, 0) = 0.$

(a) (13 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) (3 points) (TRUE or FALSE) $\lim_{t \to \infty} u(1, t) = 0.5$. Explain your answer.