This exam has 12 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.**
1. (12 points) Consider the initial value problem

\[ ty' + 2y = e^{-4t}, \quad y(1) = 0. \]

(a) (8 points) Solve the initial value problem.

(b) (2 points) Find \( \lim_{t \to \infty} y(t) \).

(c) (2 points) What is the largest interval on which the solution in (a) is certain to uniquely exist according to the Existence and Uniqueness theorem?
2. (15 points)

(a) (6 points) Suppose \( y_1(t) \) and \( y_2(t) \) are two solutions of the second order linear equation

\[
t^2 y'' + 2ty' + te^{3t}y = 0,
\]

such that \( W(y_1, y_2)(1) = 7 \). Find their Wronskian, \( W(y_1, y_2)(t) \).

(b) (5 points) Suppose it is known that \( y_1 = -3e^{6t} + 3t \cos(5t) \) and \( y_2 = 10t e^{6t} + 3t \cos(5t) \)

are two solutions of a second order linear equation

\[
y'' + by' + cy = g(t),
\]

where \( b \) and \( c \) are constants. Then what is the general solution of this equation?

(c) (4 points) Find the general solution of the third order linear equation

\[
2y''' + 7y'' = 0.
\]
3. (8 points) For each part below, construct an equation in the form of $mu'' + \gamma u' + ku = F(t)$, with your choices of $m$, $\gamma$, $k$, and $F(t)$, where appropriate, that describes a mass-spring system having the indicated property. You must clearly show that each answer does have the correct property. (Please note that the answer to neither part is unique. There are many possible correct answers for each. You just need to find one such equation for each part.)

(a) The system is underdamped, and has $\gamma = 6$.

(b) The system is undergoing resonance, and has $m = 2$.

4. (6 points) Find the Laplace transform of $f(t) = u_2(t)e^{6t}$.
5. (10 points)

(a) (8 points) Find the general solution of the system of linear equations

$$x' = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} x.$$ 

(b) (2 points) What are the type and stability of its critical point at $(0,0)$?
6. (12 points) Solve the initial value problem

\[ y'' + 4y' + 3y = \delta(t - 11), \quad y(0) = -1, \quad y'(0) = 1. \]
7. (8 points) Consider the boundary value problem

\[ u_{xx} + 8x^2u_x + 3tu_t = 0, \quad u(0, t) = 0, \quad u_x(12, t) = 0. \]

Use the substitution \( u(x, t) = X(x)T(t) \) to separate the equation into two ordinary differential equations, and rewrite each of the boundary conditions in terms of a single independent variable.

8. (8 points) For each part below, find the steady-state solution, \( v(x) \), of the following heat conduction equation given the indicated boundary conditions:

\[ 20u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0. \]

(a) \( u_x(0, t) = -2, \quad u(4, t) = 6 \)

(b) \( u(0, t) - u_x(0, t) = 0, \quad u(4, t) = 10 \)
9. (14 points) Consider the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) = 0. \]

(a) (10 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) (4 points) Is \( \lambda = 0 \) an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.
10. (25 points) Let \( f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 2, & 2 \leq x < 3 \end{cases} \)

   (a) (4 points) Consider the odd periodic extension, of period \( T = 6 \), of \( f(x) \). Sketch 3 periods, on the interval \(-9 \leq x \leq 9\), of this odd periodic extension.

   (b) (6 points) Find the Fourier coefficients \( a_1 \) and \( b_2 \) of the Fourier series of the periodic function described in (a).

   (c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -\frac{5}{2} \)? At \( x = 2 \)?

   (d) (4 points) Consider the even periodic extension, of period \( T = 6 \), of \( f(x) \). Sketch 3 periods, on the interval \(-9 \leq x \leq 9\), of this even periodic extension.

   (e) (5 points) Write down an integral formula that will give the Fourier cosine coefficients of the even periodic function described in (d).

   (f) (2 points) To what value does the Fourier series of this even periodic extension converge at \( x = 32 \)?
11. (16 points) Suppose the temperature distribution function $u(x, t)$ of a rod that has both ends perfectly insulated is given by the initial-boundary value problem

$$
6u_{xx} = u_t, \quad 0 < x < 2\pi, \quad t > 0
$$

$$
u_x(0, t) = 0, \quad u_x(2\pi, t) = 0,
$$

$$
u(x, 0) = 9 - 8\cos(4x) + 7\cos(6x).
$$

(a) (10 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) (2 points) What is $\lim_{t \to \infty} u(2, t)$?

(c) (4 points) Suppose the initial condition is, instead, $u(x, 0) = 10 - 5\sin(8x)$. How would you find the coefficients of the particular solution?
12. (16 points) Suppose the displacement $u(x,t)$ of a piece of flexible string that has both ends firmly fixed in places is given by the initial-boundary value problem

$$
\begin{align*}
9u_{xx} &= u_{tt}, & 0 < x < 5, & t > 0 \\
u(0,t) &= 0, & u(5,t) &= 0, \\
u(x,0) &= 0, \\
u_t(x,0) &= 4.
\end{align*}
$$

(a) (6 points) Based on the boundary conditions, write down the general form of the displacement $u(x,t)$.

(b) (8 points) Find all coefficients of the particular solution of this initial-boundary value problem.

(c) (2 points) What is $\lim_{t \to \infty} u(2.5,t)$?