This exam has 18 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

Please turn off and put away your cell phone.

You may not use a calculator on this exam.

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Multiple Choice Section

1. (6 points) Consider the autonomous equation

\[ y' = y^2(y - 1)(y - 3). \]

Suppose \( y(10) = \lambda \). Find all possible values of \( \lambda \) such that \( \lim_{t \to +\infty} y(t) = 1 \).

(a) \( -\infty < \lambda < 3 \)
(b) \( -\infty < \lambda \leq 1 \)
(c) \( 0 < \lambda < 3 \)
(d) \( 0 < \lambda < \infty \)

2. (6 points) Let \( y_1(t) \) and \( y_2(t) \) be any two solutions of the second order linear equation

\[ t^2 y'' - 3ty' - t \sin(4t)y = 0. \]

What is the general form of their Wronskian?

(a) \( W(y_1, y_2)(t) = \frac{C}{t^3} \)
(b) \( W(y_1, y_2)(t) = Ce^{-3t^2} \)
(c) \( W(y_1, y_2)(t) = Ct^3 \)
(d) \( W(y_1, y_2)(t) = Ce^{3t^2} \)
3. (6 points) Consider the linear system below

\[ x' = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix} x. \]

The critical point \((0,0)\) is a(n)

(a) asymptotically stable spiral point.
(b) (neutrally) stable center.
(c) unstable saddle point.
(d) asymptotically stable improper node.

4. (6 points) The point \((x, y) = (-1, 0)\) is a critical point of the nonlinear system of equations

\[
\begin{align*}
x' &= xy^2 - 2xy \\
y' &= xy + y - x - 1
\end{align*}
\]

This critical point is a(n)

(a) asymptotically stable spiral point.
(b) (neutrally) stable center.
(c) unstable saddle point.
(d) asymptotically stable improper node.
5. (6 points) Consider the fourth order linear equation
\[ y^{(4)} + 4y'' + 4y = 0. \]
What is its general solution?

(a) \[ y(t) = C_1 + C_2 t + C_3 t \cos \sqrt{2} t + C_4 t \sin \sqrt{2} t \]
(b) \[ y(t) = C_1 t + C_2 t^2 + C_3 t \cos 2t + C_4 t \sin 2t \]
(c) \[ y(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t \]
(d) \[ y(t) = C_1 \cos \sqrt{2} t + C_2 \sin \sqrt{2} t + C_3 t \cos \sqrt{2} t + C_4 t \sin \sqrt{2} t \]

6. (6 points) For which of the following equations ALL solutions approach zero as \( t \to +\infty \)?

(a) \[ y'' - 6y' + 9y = 0 \]
(b) \[ y'' + 6y' + 9y = 0 \]
(c) \[ y'' + 2y' - 3y = 0 \]
(d) \[ y'' - 2y' - 3y = 0 \]
7. (6 points) Which function is the most suitable choice of the form of particular solution $Y$ that you should use to solve the equation below?

$$y'' - 4y' + 3y = te^{-t} + 2e^{t}$$

(a) $Y = (At + B)e^{-t} + Ce^{t}$
(b) $Y = (At^2 + Bt)e^{-t} + Cte^{t}$
(c) $Y = (At^2 + Bt)e^{-t} + Ce^{t}$
(d) $Y = (At + B)e^{-t} + Cte^{t}$

8. (6 points) Which of the following is a solution of the equation

$$\begin{cases} 
    u_{xx} + u_{yy} = 0 & \text{for } x \in (0, \pi), \ y \in (0, 2) \\
    u(0, y) = 0, & u(\pi, y) = 0 \\
    u(x, 0) = 0, & u(x, 2) = \sin(2x). 
\end{cases}$$

(a) $u(x, y) = \sum_{n=1}^{\infty} \sinh(ny) \sin(nx)$
(b) $u(x, y) = \frac{1}{\sinh(4)} \sinh(2y) \sin(2x)$
(c) $u(x, y) = \sum_{n=1}^{\infty} e^{n^2 y} \sin(nx)$
(d) $u(x, y) = e^{4y} - 4 \sin(2x)$
9. (6 points) Find an implicit solution of the following equation
\[ y' = y^2 + y \]

(a) \[ y = \frac{-1 \pm \sqrt{1 - 4}}{2} \]
(b) \[ \ln |y| - \ln |1 + y| = x + C \]
(c) \[ \ln(y^2 + y) = x \]
(d) \[ \frac{-1}{y} + \ln |y| = x + C \]

10. (6 points) Find the integrating factor \( \mu(t) \) for the following equation
\[ ty' - y = \sin(t) \]

(a) \[ \mu(t) = C_1 \sin(t) + C_2 \cos(t) \]
(b) \[ \mu(t) = e^{-t} \]
(c) \[ \mu(t) = t^{-1} \]
(d) \[ \mu(t) = t \]
11. (6 points) Evaluate

\[ \mathcal{L}\{u_2(t)e^t + \delta(t-1)e^t\}(s) \]

(a) \[ \frac{e^{-2s}}{s} \frac{1}{s-1} + e^{-s} \frac{1}{s-1} \]

(b) \[ e^{-2s+2} \frac{1}{s-1} + e^{-s+1} \frac{1}{s-1} \]

(c) \[ e^{-2s+2} \frac{1}{s-1} + e^{-s+1} \]

(d) \[ e^{-2s} \frac{1}{s-1} + e^{-s} \]

12. (6 points) Evaluate

\[ \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^2 - s} \right\}(t) \]

(a) \( u_1(t)e^t \)

(b) \( u_1(t)e^{t-1} \)

(c) \( u_1(t)(e^t - 1) \)

(d) \( u_1(t)(e^{t-1} - 1) \)
13. (6 points) According to the Existence and Uniqueness Theorem, which of the following initial and boundary value problems are guaranteed to have a unique solution?

(a) \( y'' + 4y = 0, \quad y(0) = 0, \quad y(\pi) = 0 \)
(b) \( y'' + 4y = 0, \quad y(0) = 0, \quad y(\pi) = 1 \)
(c) \( y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 0 \)
(d) \( ty'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1 \)
14. (16 points) Consider the following eigenvalue–eigenfunction problem

\[ X''(x) = \lambda X(x), \quad X(0) = 0, \quad X(\pi) = 0. \]

Find all eigenvalues and corresponding eigenfunctions of the problem.
15. (16 points) Let \( f(x) = 1 \) for \( 0 < x < 2 \).

(a) (4 points) Consider the **odd** periodic extension of \( f(x) \) with period \( T = 4 \). Sketch 2 periods of this odd periodic extension on the interval \(-4 < x < 4\).

(b) (8 points) Find the Fourier series of the periodic function described in (a).

(c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -\frac{1}{2} \)? At \( x = -2 \)?
16. (16 points) Solve the one-dimensional wave problem

\[
\begin{cases}
  u_{tt} = 4u_{xx}, & 0 < x < 5, \quad t > 0 \\
  u(0, t) = 0, & u(5, t) = 0, \\
  u(x, 0) = 2 \sin(\pi x) - 3 \sin(2\pi x), \\
  u_t(x, 0) = 0.
\end{cases}
\]
17. (12 points) Let \( u(t, x) \) be the solution of
\[
\begin{cases}
  u_t = u_{xx} & 0 < x < \pi, \ t > 0 \\
  u(0, t) = -1, & u(\pi, t) = 1 \\
  u(x, 0) = -\cos(x).
\end{cases}
\]

(a) (4 points) What is the physical meaning of the boundary conditions?

(b) (8 points) Find \( \lim_{t \to +\infty} u(x, t) = \)

18. (12 points) Let \( u(t, x) \) be the solution of
\[
\begin{cases}
  u_t = u_{xx} & 0 < x < \pi, \ t > 0 \\
  u_x(0, t) = 0, & u_x(\pi, t) = 0 \\
  u(x, 0) = \cos(2x) + 1.
\end{cases}
\]

(a) (4 points) What is the physical meaning of the boundary conditions?

(b) (8 points) Find \( \lim_{t \to +\infty} u(x, t) = \)