This exam has 15 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown.** For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.**

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1. (6 points) Consider the autonomous equation
\[ y' = -\cos y. \]
Given the initial condition \( y(2) = 1 \), find \( \lim_{t \to \infty} y(t) \).

(a) 0
(b) \( \frac{\pi}{2} \)
(c) \( -\frac{\pi}{2} \)
(d) \( \pi \)

2. (6 points) Consider the second order equation
\[ y'' + \lambda y = 0, \]
where \( \lambda \) is a real number. Which statement below about the equation is false?

(a) For any value of \( \lambda \), its general solution is in the form of \( y = C_1 y_1 + C_2 y_2 \), where \( y_1 \) and \( y_2 \) are two fundamental solutions of the equation.

(b) If \( \lambda = 0 \), then its general solution is \( y(t) = C_1 + C_2 t \).

(c) For any value of \( \lambda \), there is a unique solution satisfying boundary conditions \( y(0) = 0 \) and \( y'(2\pi) = 0 \).

(d) For any value of \( \lambda \), there is a unique solution satisfying initial conditions \( y(\pi) = 0 \) and \( y'(\pi) = 32 \).
3. (6 points) Which pair of functions below can be a fundamental set of solutions for a second order homogeneous linear equation?

(a) 0, \( 4e^t + 1 \)
(b) 1, \( \cos 3t \)
(c) \( e^{-3t} \), \( e^{-3(t+5)} \)
(d) \( t^2 - 3 \), \( 9 - 3t^2 \)

4. (6 points) Evaluate the following definite integral

\[
\int_0^\infty e^{-(s+1)t} \sin(3t) \, dt.
\]

(Hint: This integral represents the Laplace transform of a certain function. It is absolutely not necessary to integrate in order to find the answer.)

(a) \( \frac{3}{s^2 + 2s + 10} \)
(b) \( e^{-s} \frac{3}{s^2 - 2s + 10} \)
(c) \( e^{-s} \frac{3}{s^2 + 2s + 10} \)
(d) \( \frac{3}{s^2 - 2s + 10} \)
5. (6 points) Find the inverse Laplace transform of

\[ F(s) = \frac{2e^{-6s}}{s(s - 2)}. \]

(a) \( f(t) = 2u_6(t)e^{-2t} \)

(b) \( f(t) = 2\delta(t - 6)(1 - e^{-2t}) \)

(c) \( f(t) = u_6(t)(1 - e^{2t+12}) \)

(d) \( f(t) = u_6(t)(e^{2t-12} - 1) \)

6. (6 points) Consider the 2 × 2 system of first order equations

\[ \mathbf{x}' = \begin{bmatrix} 2 & 5 \\ 0 & -2 \end{bmatrix} \mathbf{x}. \]

Which of the following statements is true?

(a) There is a critical point at (0, 0), which is a saddle point.

(b) All solutions approach 0 asymptotically as \( t \to \infty \).

(c) All nonzero solutions will diverge to \( \infty \) as \( t \to \infty \).

(d) The eigenvalues of the coefficient matrix are purely imaginary.
7. (6 points) Given that the point \((0,0)\) is a critical point of the nonlinear system of equations

\[
\begin{align*}
    x' &= x(1-x-y) \\
    y' &= y(1-x-y) .
\end{align*}
\]

Which of the following statements is true?

(a) This critical point is asymptotically stable.
(b) This critical point is unstable.
(c) This critical point is a center.
(d) This critical point is a saddle point.

8. (6 points) Find the steady-state solution, \(v(x)\), of the heat conduction problem with nonhomogeneous boundary conditions:

\[
\begin{align*}
    &9u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0 \\
    &2u(0,t) - 3u_x(0,t) = 0, \quad u(5,t) = 13.
\end{align*}
\]

(a) \(v(x) = 4x - 7\)
(b) \(v(x) = 2x + 3\)
(c) \(v(x) = 3x - 2\)
(d) \(v(x) = 13x\)
9. (6 points) Let \( f(x) \) be a periodic function with period \( 2\pi \), given as

\[
f(x) = 1 + 0.001 \cos(99x) - 109 \sin(5x).
\]

What are the Fourier coefficients for \( f(x) \)?

(a) \( a_0 = 1, a_1 = 0.001, a_2 = -109, \); all others are 0.

(b) \( a_0 = 2, a_1 = 0.001, a_2 = -109, \); all others are 0.

(c) \( a_0 = 2, a_{99} = 0.001, b_5 = -109, \); all others are 0.

(d) \( a_0 = 1, b_{99} = 0.001, b_5 = -109, \); all others are 0.
10. (15 points) True or false:

(a) (3 points) A suitable integrating factor that could be used to solve the linear equation

\[ ty' + (1 - 2t)y = e^{-t} \cos(2t), \quad t > 0, \]

is \( \mu(t) = te^{-2t} \).

(b) (3 points) Every exact equation \( M(x, y) + N(x, y)y' = 0 \) is also separable.

(c) (3 points) Given two solutions \( y_1(t) = \cos(4t) \) and \( y_2(t) = \sin(4t) \) of a second order homogeneous linear equation, the Wronskian \( W(y_1, y_2) = 16 \).

(d) (3 points) It is possible to separate the partial differential equation \( t^2 u_{xx} - e^x u_t = 0 \) into two ordinary differential equations.

(e) (3 points) The Fourier series of

\[ f(x) = x^3 \sin x, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x) \]

is a sine series.
11. (15 points) Consider the differential equations listed below.

1. \( y'' + y = 0 \)
2. \( y'' - 4y' + 8y = 5e^{2t} \sin 2t \)
3. \( y''' + 4y'' + 4y' = 0 \)
4. \( y'' + y'y = 10 \)
5. \( y''' - 4y' = 0 \)
6. \( y'' + 4y = \cos 2t \)

For each of parts (a) through (e) below, write down the number corresponding to the equation on the list above with the specified behavior. There is only one correct equation to each part. However, an equation may be re-used for more than one part. **You must explain your answers.**

(a) (3 points) This equation is not linear.

(b) (3 points) Every nonzero solution of this equation oscillates freely with a constant amplitude.

(c) (3 points) This equation describes a mass-spring system that exhibits resonance.

(d) (3 points) This equation has general solution \( y = C_1 + C_2e^{-2t} + C_3te^{-2t} \).

(e) (3 points) The solution to the corresponding homogeneous equation of this equation is \( y_c = C_1 \cos 2t + C_2 \sin 2t \).
12. (16 points) Consider the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(5) = 0. \]

(a) (12 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) (4 points) Is \( \lambda = 0 \) an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.
13. (19 points) Let \( f(x) = 2 + \sin x \), \( 0 < x < \pi \).

(a) (4 points) Consider the odd periodic extension, of period \( T = 2\pi \), of \( f(x) \). Sketch 3 periods, on the interval \(-3\pi < x < 3\pi\), of this odd periodic extension.

(b) (5 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

(i) \[ b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 + \sin x) \sin nx \, dx \]

(ii) \[ b_n = \frac{2}{\pi} \int_{0}^{\pi} (2 + \sin x) \sin nx \, dx \]

(iii) \[ b_n = \frac{1}{2\pi} \int_{0}^{\pi} (2 + \sin x) \sin \frac{nx}{2} \, dx \]

(iv) \[ b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} (2 + \sin x) \sin \frac{nx}{2} \, dx \]

(c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -\pi \)? At \( x = \frac{3\pi}{2} \)?

(d) (4 points) Consider the even periodic extension, of period \( T = 2\pi \), of \( f(x) \). Sketch 3 periods, on the interval \(-3\pi < x < 3\pi\), of this even periodic extension.

(e) (2 points) To what value does the Fourier series of this even periodic extension converge at \( x = \pi \)?
14. (14 points) Suppose the displacement $u(x, t)$ of a piece of flexible string of length $L$ that has both ends firmly fixed in places is given by the initial-boundary value problem

$$
100u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\
u(0, t) = 0, \quad u(L, t) = 0, \\
u(x, 0) = f(x), \\
u_t(x, 0) = 0.
$$

(a) (3 points) TRUE or FALSE: At $t = 0$, the string is at rest (i.e., having zero initial velocity).

(b) (5 points) Based on the boundary conditions, write down the general form of the displacement $u(x, t)$.

(c) (6 points) Write down, but do not evaluate, the integral(s) that would be used to find all coefficients of the particular solution of this initial-boundary value problem.
15. (17 points) Suppose the temperature distribution function \( u(x, t) \) of a rod that has both ends kept at different temperatures is given by the initial-boundary value problem

\[
3u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0
\]
\[
u(0, t) = 32, \quad u(4, t) = 52,
\]
\[
u(x, 0) = 5x + 32 - 40 \sin\left(\frac{\pi}{2}x\right) - 24 \sin(2\pi x).
\]

(a) (3 points) What is its steady-state solution?

(b) (10 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(c) (2 points) What is \( \lim_{t \to \infty} u(2, t) \)?

(d) (2 points) Suppose the initial condition is, instead, \( u(x, 0) = 2x + 40 + 60 \sin(10x) \). Will the limit, \( \lim_{t \to \infty} u(2, t) \), be higher than, lower than, or equal to the temperature you found in part (c)?