MATH 251
Final Examination
December 14, 2011
FORM A

Name:_____________________
Student Number:_____________
Section:__________________

This exam has 18 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.

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14:__________
15:__________
16:__________
17:__________
18:__________
Total:__________
1. (6 points) Consider the autonomous equation

\[ y' = (y - 6)y^2. \]

Given the initial condition \( y(10) = 3 \), find \( \lim_{t \to \infty} y(t) \).

(a) \(-\infty\)
(b) 0
(c) 6
(d) \(\infty\)

2. (6 points) Given that the equation below is an exact equation.

\[ 4x^3y^3 - 2y + (\alpha x^4 y^2 - 2x + 2y) y' = 0 \]

Find the value of the coefficient \( \alpha \).

(a) \( \alpha = 1 \)
(b) \( \alpha = 2 \)
(c) \( \alpha = 3 \)
(d) \( \alpha = 4 \)
3. (6 points) Suppose it is known that $y_1 = 6e^{3t} + 5t^2e^t$ and $y_2 = -\sqrt{2}e^{-2t} + 5t^2e^t$ are two solutions of a second order linear equation

$$y'' + by' + cy = g(t),$$

where $b$ and $c$ are constants. Then what is the general solution of this equation?

(a) $y = C_1e^{3t} + C_2e^{-2t} + 10t^2e^t$
(b) $y = C_1(6e^{3t} + 5t^2e^t) + C_2(-\sqrt{2}e^{-2t} + 5t^2e^t)$
(c) $y = C_1e^{3t} + C_2e^{-2t} + C_3t^2e^t$
(d) $y = C_1e^{3t} + C_2e^{-2t} + 5t^2e^t$

4. (6 points) Consider the two initial / boundary value problems below. Which is certain to have a unique solution for every value of $\alpha$?

(I) $y'' + 9y = 0, \quad y(\alpha) = \alpha^2, \quad y'(\alpha) = -\alpha.$

(II) $y'' + 9y = 0, \quad y(0) = 0, \quad y'(\alpha^2) = 0.$

(a) I only.
(b) II only.
(c) Both I and II.
(d) Neither.
5. (6 points) Let \( y_1(t) \) and \( y_2(t) \) be any two solutions of the second order linear equation
\[
3t^2 y'' + 6ty' + e^{-5t}y = 0.
\]

What is the general form of their Wronskian, \( W(y_1, y_2)(t) \)?

(a) \( Ct^{-2} \)
(b) \( Ce^{3t^2} \)
(c) \( Ce^{-3t^2} \)
(d) \( Ct^2 \)

6. (6 points) Suppose the mass-spring system described by the equation below has displacement, \( u(t) \), whose amplitude increases proportionally with time \( t \). Find the value of \( k \).
\[
6u'' + ku = 36 \sin 3t
\]

(a) \( k = 9 \)
(b) \( k = 18 \)
(c) \( k = 36 \)
(d) \( k = 54 \)
7. (6 points) Find the Laplace transform of \( f(t) = u_6(t)e^{-3t} \).

(a) \( F(s) = e^{-6s+18}\frac{-6s - 17}{(s + 3)^2} \)

(b) \( F(s) = e^{-6s-18}\frac{6s + 19}{(s + 3)^2} \)

(c) \( F(s) = e^{-6s}\frac{1}{s(s + 3)^2} \)

(d) \( F(s) = e^{-6s}\frac{1}{(s + 3)^2} \)

8. (6 points) Find the inverse Laplace transform of

\[
F(s) = \frac{e^{-2s}(3s + 2)}{s^2 + 16}.
\]

(a) \( f(t) = u_2(t)(3 \cos(4t) + 2 \sin(4t)) \)

(b) \( f(t) = u_2(t)(3 \cos(4t - 2) + 2 \sin(4t - 2)) \)

(c) \( f(t) = u_2(t)(3 \cos(4t + 8) + \frac{1}{2} \sin(4t + 8)) \)

(d) \( f(t) = u_2(t)(3 \cos(4t - 8) + \frac{1}{2} \sin(4t - 8)) \)
9. (6 points) Consider all the nonzero solutions of the linear system
\[ \mathbf{x}' = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \mathbf{x}. \]
As \( t \to \infty \),
(a) some converge to \((0,0)\), the others move away, unbounded, from \((0,0)\).
(b) they all converge to \((0,0)\).
(c) they all move away, unbounded, from \((0,0)\).
(d) they neither converge to \((0,0)\), nor move away unbounded from \((0,0)\).

10. (6 points) Given that the point \((-1,4)\) is a critical point of the nonlinear system of equations
\[
\begin{align*}
x' &= xy - 2x + y - 2 \\
y' &= xy - 4x
\end{align*}
\]
This critical point \((-1,4)\) is a(n)
(a) asymptotically stable spiral point.
(b) unstable node.
(c) unstable saddle point.
(d) asymptotically stable improper node.
11. (6 points) Consider the two linear partial differential equations.

\[(I)\] \[u_{xx} + 4u_{xt} + 4u_{tt} = 0\]

\[(II)\] \[u_{xx} - 4u_{tx} - 5u = 0\]

Use the substitution \(u(x, t) = X(x)T(t)\) and attempt to separate each equation into two ordinary differential equations. Which statement below is true?

(a) Neither equation is separable.
(b) Only (I) is separable.
(c) Only (II) is separable.
(d) Both equations are separable.

12. (6 points) Find the steady-state solution, \(v(x)\), of the heat conduction problem with nonhomogeneous boundary conditions:

\[\alpha^2 u_{xx} = u_t, \quad 0 < x < 6, \quad t > 0\]

\[u(0, t) + 2u_x(0, t) = 5, \quad u(6, t) - 5u_x(6, t) = 1, \quad u(x, 0) = f(x).\]

(a) \(v(x) = \frac{4}{5} x + 5\)
(b) \(v(x) = \frac{-4}{5} x + 1\)
(c) \(v(x) = 3x - 4\)
(d) \(v(x) = 4x - 3\)
13. (6 points) Consider the Fourier series (of period $6\pi$) representing

$$f(x) = 5x^3 + 7, \quad -3\pi < x < 3\pi, \quad f(x + 6\pi) = f(x).$$

Which statement below is true?

(a) The Fourier series is a cosine series.
(b) The Fourier series is a sine series.
(c) The Fourier series is neither a cosine series nor a sine series.
(d) The function does not have a Fourier series because it is not periodic.

14. (6 points) Consider the wave equation initial-boundary value problem

$$9u_{xx} = u_{tt}, \quad 0 < x < 2, \quad t > 0$$
$$u(0, t) = 0, \quad u(2, t) = 0,$$
$$u(x, 0) = 0,$$
$$u_t(x, 0) = g(x).$$

Which function below could be one of its solutions?

(a) $u(x, t) = \sin \frac{3\pi t}{2} \sin \frac{\pi x}{2}$
(b) $u(x, t) = 3 \sin \frac{9\pi x}{2} \sin \frac{3\pi t}{2}$
(c) $u(x, t) = 5 \cos \frac{15\pi t}{2} \sin \frac{5\pi x}{2}$
(d) $u(x, t) = 7 \cos \frac{21\pi x}{2} \sin \frac{7\pi t}{2}$
15. (15 points) True or false:

(a) (3 points) The equation \( y' = \frac{t}{y} \) is an example of a first order equation that is separable but is not also linear.

(b) (3 points) The general solution of the equation \( 4y'' + 4y' + y = 0 \) is 
\[ y = C_1e^{t/2} + C_2te^{t/2}. \]

(c) (3 points) Using the formula \( u(x,t) = X(x)T(t) \), the boundary conditions \( u_x(0,t) = 0 \) and \( u(1,t) = 1 \) can be rewritten as \( X'(0) = 0 \) and \( X(1) = 1 \).

(d) (3 points) Let \( A \) be a positive constant. When appearing with the heat conduction equation, \( \alpha^2 u_{xx} = u_t, \ 0 < x < L \), the boundary conditions \( u(0,t) = A \) and \( u(L,t) = 2A \), mean that the temperature at the left end of the rod is twice as high as the temperature at the right end.

(e) (3 points) The constant term of the Fourier series representing 
\[ f(x) = 2x^3, \quad -1 < x < 1, \quad f(x+2) = f(x), \]

is zero.
16. (16 points) Consider the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(\pi) = 0. \]

(a) (12 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) (4 points) Is \( \lambda = 0 \) an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.
17. (19 points) Let \( f(x) = 2x, \quad 0 < x < 1. \)

(a) (4 points) Consider the **odd** periodic extension, of period \( T = 2, \) of \( f(x) \). Sketch 3 periods, on the interval \(-3 < x < 3, \) of this odd periodic extension.

(b) (5 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

\[
\begin{align*}
(i) \quad b_n &= 2 \int_{0}^{1} x \sin \left( \frac{n\pi x}{2} \right) \, dx \\
(ii) \quad b_n &= 4 \int_{0}^{1} x \sin(n\pi x) \, dx \\
(iii) \quad b_n &= \frac{1}{2} \int_{-1}^{1} x \sin \left( \frac{n\pi x}{2} \right) \, dx \\
(iv) \quad b_n &= \int_{-1}^{1} x \sin(n\pi x) \, dx
\end{align*}
\]

(c) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = \frac{-3}{2} \)? At \( x = \frac{1}{2} \)?

(d) (4 points) Consider the **even** periodic extension, of period \( T = 2, \) of \( f(x) \). Sketch 3 periods, on the interval \(-3 < x < 3, \) of this even periodic extension.

(e) (2 points) To what value does the Fourier series of this even periodic extension converge at \( x = 11 \)?
18. (16 points) Suppose the temperature distribution function \( u(x, t) \) of a rod that has both ends perfectly insulated is given by the initial-boundary value problem

\[
3u_{xx} = u_t, \quad 0 < x < 2\pi, \quad t > 0
\]
\[
u_x(0, t) = 0, \quad u_x(2\pi, t) = 0,
\]
\[
u(x, 0) = 16 - 25\cos(2x) - 36\cos(5x).
\]

(a) (12 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) (2 points) What is \( \lim_{t \to \infty} u(\pi, t) \)?

(c) (2 points) Suppose the initial condition is, instead, \( u(x, 0) = 32 - 100\cos(9x) \). Will the limit, \( \lim_{t \to \infty} u(\pi, t) \), be higher than, lower than, or equal to the temperature you found in part (b)?