This exam has 18 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown.** For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**Please turn off and put away your cell phone.**

**You may not use a calculator on this exam.**

<table>
<thead>
<tr>
<th>Do not write in this box.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>14:</td>
</tr>
<tr>
<td>15:</td>
</tr>
<tr>
<td>16:</td>
</tr>
<tr>
<td>17:</td>
</tr>
<tr>
<td>18:</td>
</tr>
<tr>
<td>Total:</td>
</tr>
</tbody>
</table>
1. (6 points) Find the solution of the initial value problem

\[ y' = \frac{\cos t}{\sin y}, \quad y(0) = \frac{\pi}{2}. \]

(a) \( y = \cos^{-1}(-\sin t) \)
(b) \( y = \cos^{-1}(\sin t - 1) \)
(c) \( y = -\sin^{-1}(\cos t) \)
(d) \( y = \sin^{-1}(-\cos t + \frac{\pi}{2}) \)

2. (6 points) Which initial or boundary value problem below is guaranteed to have a unique solution according to the Existence and Uniqueness theorems?

(a) \( y'' + \sin(5t)y' - \cos(10t)y = \pi, \quad y(0) = 1, \quad y'(0) = -1. \)
(b) \( (t + 2)y' - e^{-t}y = t, \quad y(-2) = 0. \)
(c) \( y'' + 100y = 0, \quad y(0) = 9, \quad y(2\pi) = -10. \)
(d) \( t^2y'' + ty' + y = 0, \quad y(0) = -2, \quad y'(0) = 3. \)
3. (6 points) Let \( y_1(t) \) and \( y_2(t) \) be any two solutions of the second order linear equation
\[
2ty'' + 4y' - t^3 \cot(2t)y = 0.
\]
What is the general form of their Wronskian, \( W(y_1, y_2)(t) \)?

(a) \( Ct^4 \)
(b) \( Ce^{-4t} \)
(c) \( Ce^{2t} \)
(d) \( Ct^{-2} \)

4. (6 points) Which of the functions below is a particular solution of the nonhomogeneous linear equation
\[
y'' + 3y' + 2y = 2t + 1?
\]

(a) \( y = 2e^{-t} \)
(b) \( y = t - 1 \)
(c) \( y = -11e^{-2t} \)
(d) \( y = 4e^{-t} + e^{-2t} + 2t + 1 \)
5. (6 points) A certain mass-spring system is described by the equation

\[ 3u'' + \gamma u' + 12u = 0. \]

Find all values of \( \gamma \) such that the system would be underdamped.

(a) \(-12 < \gamma < 12\)
(b) \(0 < \gamma < 12\)
(c) \(\gamma \leq 144\)
(d) \(0 \leq \gamma < 144\)

6. (6 points) Find the general solution of the fourth order linear equation

\[ y^{(4)} + 3y^{(3)} - 4y' = 0. \]

(a) \(y(t) = C_1 \cos t + C_2 \sin t + C_3 e^t + C_4 e^{-4t}\)
(b) \(y(t) = C_1 t + C_2 t^2 + C_3 e^{-t} + C_4 e^{4t}\)
(c) \(y(t) = C_1 + C_2 t + C_3 e^t + C_4 e^{-4t}\)
(d) \(y(t) = C_1 e^{-t} + C_2 e^{4t} + C_3 te^{-t} + C_4 te^{4t}\)
7. (6 points) The inverse Laplace transform of \( F(s) = e^{-4s} \frac{s - 1}{s^2 + 7s + 10} \) is

(a) \( f(t) = u_4(t)(e^{-2t+8} - 2e^{-5t+20}) \)
(b) \( f(t) = u_4(t)(e^{-2t-8} - 2e^{-5t-20}) \)
(c) \( f(t) = u_4(t)(-e^{-2t-8} + 2e^{-5t-20}) \)
(d) \( f(t) = u_4(t)(-e^{-2t+8} + 2e^{-5t+20}) \)

8. (6 points) Consider the linear system below.

\[
\begin{bmatrix}
8 & 0 \\
0 & 8
\end{bmatrix} \begin{bmatrix}
x \\
x'
\end{bmatrix}.
\]

Which statement below is true?

(a) The critical point \((0, 0)\) is an asymptotically stable proper node.
(b) The critical point \((0, 0)\) is an unstable improper node.
(c) \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{8t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) is a general solution of the system.
(d) \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} t \\ t+1 \end{bmatrix} \) is a general solution of the system.
9. (6 points) Given that the point $(2, -2)$ is a critical point of the nonlinear system of equations

\[
\begin{align*}
    x' &= x^2 - y^2 \\
    y' &= xy + 2x - y - 2
\end{align*}
\]

This critical point $(2, -2)$ is an

(a) unstable node.
(b) unstable saddle point.
(c) asymptotically stable improper node.
(d) asymptotically stable spiral point.

10. (6 points) Consider the two linear partial differential equations.

\[
\begin{align*}
    (I) & \quad u_{xx} = 2u_t + 1 \\
    (II) & \quad u_{xx} = 2u_t + 3u
\end{align*}
\]

Use the substitution $u(x, t) = X(x)T(t)$ and attempt to separate each equation into two ordinary differential equations. Which statement below is true?

(a) Neither equation is separable.
(b) Only (I) is separable.
(c) Only (II) is separable.
(d) Both equations are separable.
11. (6 points) Find the Fourier cosine coefficient corresponding to \( n = 2, \ a_2, \) of the Fourier series of period \( T = 2\pi \) representing the function \( f(x) = 6\cos 2x. \)

(a) \( a_2 = 0 \)
(b) \( a_2 = 6 \)
(c) \( a_2 = \frac{3}{\pi} \)
(d) \( a_2 = -\frac{1}{\pi} \)

12. (6 points) Consider the Fourier series (of period 20) representing

\[ f(x) = x^5, \quad -10 < x < 10, \quad f(x + 20) = f(x). \]

Which statement below is true?

(a) The Fourier series is a cosine series.
(b) The Fourier series is a sine series.
(c) The Fourier series is neither a cosine series nor a sine series.
(d) The function does not have a Fourier series because it is not periodic.
13. (6 points) Find the steady-state solution, $v(x)$, of the heat conduction problem with nonhomogeneous boundary conditions:

\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0 \]
\[ u(0, t) + u_x(0, t) = 0, \quad u(5, t) = 8, \]
\[ u(x, 0) = f(x) = 10x + 8. \]

(a) $v(x) = \frac{8}{5}x$
(b) $v(x) = 10x + 8$
(c) $v(x) = -x + 13$
(d) $v(x) = 2x - 2$

14. (6 points) Which function below is a solution of the wave equation boundary value problem

\[ 4u_{xx} = u_{tt}, \quad u(0, t) = 0, \quad u(3, t) = 0. \]

(a) $u(x, t) = 5 \cos(4\pi t) \sin(2\pi x)$
(b) $u(x, t) = t - \sin(9\pi x)$
(c) $u(x, t) = e^{-4\pi^2 t} \cos(\pi x)$
(d) $u(x, t) = 6e^{4xt} \sin(3\pi x)$
15. (14 points) Use the Laplace transform to solve the following initial value problem.

\[ y'' + 36y = 3\delta(t - 1), \quad y(0) = -4, \quad y'(0) = 0. \]

No credit will be given if the Laplace transform is not used to solve this problem.
16. (16 points) Consider the two-point boundary value problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(\pi) = 0. \]

Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem.
17. (18 points) Let \( f(x) = 4, \quad 0 < x < 2. \)

(a) (4 points) Consider the odd periodic extension, of period \( T = 4, \) of \( f(x) \). Sketch 3 periods, on the interval \(-6 < x < 6,\) of this odd periodic extension.

(b) (5 points) Find \( a_{10} \), the tenth cosine coefficient of the Fourier series of the periodic function described in (a).

(c) (5 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

\[
(i) \quad b_n = \frac{1}{2} \int_{0}^{2} \sin \frac{n \pi x}{4} \, dx \\
(ii) \quad b_n = 2 \int_{-2}^{2} \sin \frac{n \pi x}{4} \, dx \\
(iii) \quad b_n = 2 \int_{0}^{2} \sin \frac{n \pi x}{2} \, dx \\
(iv) \quad b_n = 4 \int_{0}^{2} \sin \frac{n \pi x}{2} \, dx
\]

(d) (4 points) To what value does the Fourier series of this odd periodic extension converge at \( x = -2? \) At \( x = 5? \)
18. (18 points) Suppose the temperature distribution function $u(x, t)$ of a rod that has both ends kept at different temperatures is given by the initial-boundary value problem

$$
4u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0
$$

$$
u(0, t) = 60, \quad u(1, t) = 40,
$$

$$
u(x, 0) = 60 - 20x - 30\sin(2\pi x) + 50\sin(7\pi x).
$$

(a) (4 points) Find the steady-state solution of the above initial-boundary value problem.

(b) (10 points) Based on the given boundary conditions, state the general form of the solution. Then find the particular solution of the initial-boundary value problem.

(c) (2 points) What is $\lim_{t \to \infty} u(0.9, t)$?

(d) (2 points) Suppose the initial condition is, instead, $u(x, 0) = 2000 + 30x$. Will the limit $\lim_{t \to \infty} u(0.9, t)$ be higher than, lower than, or equal to the temperature you found in part (c)?