There are 12 multiple choice questions and 5 partial credit questions. In order to obtain full credit for the partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work on a partial credit problem. The use of calculators is not permitted in this examination.

For multiple choice problems, write the letter of your choice in the space provided below.

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1. (5 points) Which of the following has a unique solution on the whole interval \((0, \pi)\)?

(a) \(y'' + y = 0, \ y(0) = 0, \ y(\pi) = 0\).
(b) \(y'' + 4y = 0, \ y'(0) = 0, \ y'(\pi) = 0\).
(c) \((t + 1)y'' + ty = 0, \ y(1) = 1, \ y'(1) = 0\).
(d) \((t - 1)y' + 2y = 0, \ y(0) = 0, \ y'(0) = 1\).

2. (5 points) Let \(y_1(t) = 1\) and \(y_2(t) = 0\). Which of the following three statements is true?

(a) \(y_1(t)\) and \(y_2(t)\) are linearly independent.
(b) \(\mathcal{L}[y_1(t)] = 1/s, \text{ for } s > 0\).
(c) \(y_2(t)\) is the unique solution of \(y' = y^{1/3}, \ y(0) = 0\).
(d) All three statements are false.

3. (5 points) For which of the following equations is it true that ALL solutions approach zero as \(t \to \infty\)?

(a) \(y'' + 2y' + y = 0\).
(b) \(y'' - 2y' + y = 0\).
(c) \(y'' + y = 0\).
(d) \(y'' - y = 0\).
4. (5 points) Consider the $2\pi$-periodic function

$$f(x) = |x|, \quad \text{when} \quad -\pi < x < \pi, \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Find the fourth sine coefficient $b_4$ of the Fourier series.

(a) $\frac{1}{4}$.

(b) $-\frac{1}{4}$.

(c) 0.

(d) $-\frac{1}{2}$.

5. (5 points) Suppose

$$v(x) = 5 - x,$$

and $v(x)$ is the steady-state solution of a heat conduction problem for a rod of length 10 cm. Which one of the following statements describes the boundary conditions?

(a) The left end is insulated and the right end is held at a constant temperature.

(b) Both ends are held at constant temperatures.

(c) The left end is held at a constant temperature and the right end is insulated.

(d) Both ends are insulated.

6. (5 points) Let $u(x, y)$ be the solution of the Laplace’s equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 7, \quad 0 < y < 3,$$

$$u(x, 0) = 0, \quad u_y(x, 3) = 0, \quad u(0, y) = 0, \quad u(7, y) = \sin(\pi y).$$

Suppose it has the following form $u(x, y) = X(x)Y(y)$. Then $X(x)$ or $Y(y)$ satisfies one of the following pairs of boundary conditions. Find the pair.

(a) $X(0) = 0, X(7) = 0$.

(b) $X(0) = 0, X(3) = 0$.

(c) $Y(0) = 0, Y'(7) = 0$.

(d) $Y(0) = 0, Y'(3) = 0$. 

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7. (5 points) Consider a $2 \times 2$ linear system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 4 \\ 0 & b \end{pmatrix},$$

where $b$ is some number. Which of the following is not true?

(a) We can choose $b$, so that the origin is an asymptotically stable node.

(b) We can choose $b$, so that the origin is an unstable node.

(c) We can choose $b$, so that the origin is a saddle.

(d) We can choose $b$, so that the origin is an improper node.

8. (5 points) $Y_1(t)$ is a solution to the equation $y'' + py' + qy = e^t$. $Y_2(t)$ is a solution to the equation $y'' + py' + qy = 3e^t$. Which of the following is a solution to the equation $y'' + py' + qy = 2e^t$?

(a) $y = Y_1(t) + Y_2(t)$.

(b) $y = Y_1(t) - Y_2(t)$.

(c) $y = 2Y_1(t)$.

(d) $y = 2Y_2(t)$.

9. (5 points) The displacement $u(x,t)$ of a vibrating string of length $\pi$ cm satisfies the following wave equation and boundary conditions:

$$\begin{cases} u_{tt} = u_{xx}, \\ u(0,t) = 0, \ u(\pi, t) = 0, \quad t > 0, \\ u(x,0) = 0, \ u_t(x,0) = 4 \sin(2x), \quad 0 < x < \pi. \end{cases}$$

Which of the following three statements is not true?

(a) The ends of the string are fixed.

(b) The string is set in motion with no initial velocity.

(c) $u(x,t) = 2 \sin(2t) \sin(2x)$.

(d) All three statements are true.
10. (5 points) Consider the predator-prey system of equations

\[ \frac{dx}{dt} = x(1 - y), \quad \frac{dy}{dt} = y(x - 1), \quad x(0) = 1, \quad y(0) = 2. \]

Which of the following three statements is true?

(a) Predator and prey populations become extinct as \( t \to \infty \).
(b) Predator and prey populations both approach the same value 1 as \( t \to \infty \).
(c) Predator and prey populations vary periodically as \( t \to \infty \).
(d) All three statements are false.

11. (5 points) The point \((3, 3)\) is a critical point of the nonlinear system of equations

\[ \frac{dx}{dt} = 3 - y, \quad \frac{dy}{dt} = -4x + y + x^2. \]

This critical point is a(n)

(a) unstable saddle point
(b) spiral, but its stability cannot be determined.
(c) unstable node.
(d) unstable spiral.

12. (5 points) Consider a nonlinear system \( \frac{dx}{dt} = 1 - y^2, \quad \frac{dy}{dt} = xy + 2y. \)

Which of the following three statements is not true?

(a) This system has exactly two critical points.
(b) \((x(t), y(t)) = (-2, 1)\) is a solution of this equation.
(c) \((x(t), y(t)) = (t, 0)\) is a solution of this equation.
(d) All three statements are true.
13. (15 points) Consider the ordinary differential equation \( y' = y - y^2 \).
   
   (a) (2 points) Find all its equilibrium solutions.

   (b) (4 points) Determine the stability of each equilibrium solutions.

   (c) (9 points) Find all solutions of the ordinary differential equation. You may leave your answer in the implicit form.
14. (15 points) Consider the ordinary differential equation $ty'' - y' + (1 - t)y = 0, \ t > 0$.

(a) (4 points) Verify that $y_1(t) = e^t$ is a solution of this equation.

(b) (8 points) Find $y_2(t)$, such that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of the ODE.

(c) (3 points) Find the general solution of the ODE.
15. (20 points) Suppose that the heat distribution in a rod is governed by the equation

\[ u_t = 4u_{xx}, \quad 0 < x < 6, \; t > 0. \]

(a) (5 points) Assume the initial distribution of temperature \( u(x, 0) = \cos \frac{3\pi x}{10} \). Also, assume that the temperature of the left end is fixed at 5 degrees, and the temperature of the right end is fixed at 15 degrees. Write down the corresponding boundary value problem. **Do not solve it.**

(b) (5 points) Find \( \lim_{t \to \infty} u(3, t) \) for the solution of part a).

(c) (10 points) Assume the initial distribution of temperature \( u(x, 0) = 10 - 2\cos(\pi x) \), and the ends are insulated. Write down the boundary value problem and solve it.
16. (20 points) Let \( f(x) = 1 + \sin x \), \( 0 \leq x < \pi \).

(a) (3 points) Sketch the graph of the odd \( 2\pi \)-periodic extension over two periods.

(b) (2 points) Find the value to which the Fourier series of the odd extension converges at \( x = 0 \).

(c) (3 points) Sketch the graph of the even \( 2\pi \)-periodic extension over two periods.

(d) (2 points) Find the value to which the Fourier series of the even extension converges at \( x = 0 \).

(e) (5 points) Find the constant term of the Fourier series of this even extension.

(f) (5 points) Write formulas for the Fourier coefficients of the non-constant terms of this even extension. Express your answer in terms of an integral starting at \( x = 0 \) but do not evaluate it.
17. (20 points) Consider $X'' + \lambda X = 0$, $X(0) = 0$, $X'(\pi/2) = 0$.

(a) (4 points) Write down a general solution of the above ODE when $\lambda = 0$. Is $\lambda = 0$ an eigenvalue? If it is, then what is the corresponding eigenfunction? If it is not, then explain why not.

(b) (8 points) Write down a general solution of the above ODE when $\lambda > 0$. Determine all eigenvalues and eigenfunctions with $\lambda > 0$.

(c) (8 points) Assume $u(x, t) = X(x)T(t)$ is a solution of the PDE

$$u_{xx} + 5u_t + 4u_{tt} = 0, \ 0 < x < \pi/2, \ t > 0.$$  

Use the method of separation of variables to obtain ODEs for $X(x)$ and $T(t)$. 