This exam has 17 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. The point value for each question is in parentheses to the right of the question number. A list of Laplace transforms is attached as the last page of this booklet. It can be removed for easy reference during the examination.

You may not use a calculator on this exam. Please turn off and put away your cell phone.
1. (6 points) Suppose $y(t)$ is the solution of the initial value problem
\[ y' = 25 - y^2, \quad y(6) = -1. \]
What is $\lim_{t \to \infty} y(t)$?

(a) $-5$.
(b) $5$.
(c) $\infty$.
(d) $-\infty$.

2. (6 points) Consider all the nonzero solutions of the second order linear equation
\[ y'' + 8y' + 16y = 0. \]
As $t \to \infty$, they will

(a) approach 0.
(b) approach $\infty$.
(c) approach $-\infty$.
(d) some approach $\infty$, while others approach $-\infty$. 

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3. (6 points) Let $y_1(t)$ and $y_2(t)$ be any two solutions of the second order linear equation

$$t^2 y'' - 6ty' + \cos(3t)y = 0.$$ 

What is the general form of their Wronskian, $W(y_1, y_2)(t)$?

(a) $Ce^{-6t}$  
(b) $Ce^{3t^2}$  
(c) $Ct^6$  
(d) $\frac{C}{t^6}$

4. (6 points) Consider the problems below.

$$(I) \quad y'' - 4y = 0, \quad y(0) = \alpha, \quad y'(0) = \beta.$$ 

$$(II) \quad y'' - 4y = 0, \quad y'(0) = \alpha, \quad y'(10) = \beta.$$ 

(a) Only $(I)$ has a unique solution for every combination of real numbers $\alpha$ and $\beta$. 
(b) Only $(II)$ has a unique solution for every combination of real numbers $\alpha$ and $\beta$. 
(c) They each has a unique solution for every combination of real numbers $\alpha$ and $\beta$. 
(d) Neither is guaranteed to have a unique solution for every combination of real numbers $\alpha$ and $\beta$. 

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5. (6 points) Which equation below describes a mass-spring system that is undergoing resonance?

(a) \( y'' + 16y = -2 \sin 4t \)

(b) \( y'' + 9y = 5 \cos 9t \)

(c) \( y'' + 4y' + 4y = \cos 2t \)

(d) \( y'' + 100y = 0 \)

6. (6 points) The inverse Laplace transform of \( F(s) = \frac{s - 3}{s^2 - 4s + 20} \) is

(a) \( e^{-2t} \cos 4t - 3e^{-2t} \sin 4t \),

(b) \( e^{-2t} \cos 4t - \frac{3}{4} e^{-2t} \sin 4t \),

(c) \( e^{2t} \cos 4t - \frac{1}{4} e^{2t} \sin 4t \),

(d) \( e^{2t} \cos 4t - e^{2t} \sin 4t \).
7. (6 points) Suppose \( y(t) \) is the solution of the initial value problem

\[
y'' + 5y' - 6y = u(t - \pi), \quad y(0) = 0, \quad y'(0) = -1.
\]

What is \( Y(s) \), the Laplace transform of \( y(t) \)? (Recall: \( u_\pi(t) = u(t - \pi) \).)

(a) \( Y(s) = \frac{e^{\pi s}}{s(s^2 + 5s - 6)} + \frac{5}{s^2 + 5s - 6} \)

(b) \( Y(s) = \frac{e^{-\pi s}}{s(s^2 + 5s - 6)} - \frac{1}{s^2 + 5s - 6} \)

(c) \( Y(s) = \frac{e^{-\pi s}}{s(s^2 + 5s - 6)} - \frac{s + 5}{s^2 + 5s - 6} \)

(d) \( Y(s) = \frac{e^{\pi s} - 1}{s^2 + 5s - 6} \)

8. (6 points) Consider the fourth order linear equation

\[
y^{(4)} + 8y'' + 16y = 0,
\]

which has a characteristic equation \( r^4 + 8r^2 + 16 = (r^2 + 4)^2 = 0 \).

What is its general solution?

(a) \( y(t) = C_1 e^{2t} + C_2 te^{2t} + C_3 e^{-2t} + C_4 te^{-2t} \)

(b) \( y(t) = C_1 t \cos 2t + C_2 t \sin 2t \)

(c) \( y(t) = C_1 e^{-2t} + C_2 te^{-2t} + C_3 \cos 2t + C_4 \sin 2t \)

(d) \( y(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t \)
9. (6 points) The point (1, 1) is a critical point of the nonlinear system of equations

\[
\begin{align*}
    x' &= 2x - 2y \\
y' &= xy + 2x - y - 2
\end{align*}
\]

This critical point is a(n)

(a) unstable saddle point.
(b) unstable spiral point.
(c) asymptotically stable node.
(d) (neutrally) stable center.

10. (6 points) Find the Fourier cosine coefficient corresponding to \( n = 4 \), \( a_4 \), of the Fourier series (period \( T = 2\pi \)) representing the function \( f(x) = \cos 4x, \ -\pi \leq x \leq \pi \).

(a) \( a_4 = 0 \)
(b) \( a_4 = 1 \)
(c) \( a_4 = \frac{-1}{2\pi} \)
(d) \( a_4 = \frac{1}{4\pi} \)
11. (6 points) Find the steady-state solution, \( v(x) \), of the heat conduction problem

\[
3u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0
\]

\[
u(0, t) = 20, \quad u(5, t) = 50,
\]

\[
u(x, 0) = f(x).
\]

(a) \( v(x) = 5x + 35 \)
(b) \( v(x) = -6x + 50 \)
(c) \( v(x) = \frac{1}{5}x + 30 \)
(d) \( v(x) = 6x + 20 \)

12. (6 points) Consider the wave equation initial-boundary value problem

\[
4u_{xx} = u_{tt}, \quad 0 < x < 6, \quad t > 0
\]

\[
u(0, t) = 0, \quad u(6, t) = 0,
\]

\[
u(x, 0) = 0, \quad u_t(x, 0) = g(x).
\]

In what specific form will its general solution appear?

(a) \( u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi t}{3} \sin \frac{n\pi x}{6} \)
(b) \( u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi t}{3} \sin \frac{n\pi x}{6} \)
(c) \( u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{3} \sin \frac{n\pi x}{6} \)
(d) \( u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi t}{3} \sin \frac{n\pi x}{6} \)
13. (18 points) True or false:
   (a) (3 points) The equation \((1 - 3x^2y^3) - 3x^3y^2 y' = 0\) is an exact equation.
   (b) (3 points) The functions \(y_1 = 1\) and \(y_2 = t\) can be a set of fundamental solutions for some second order linear differential equation, \(-\infty < t < \infty\).
   (c) (3 points) It is possible for a solution of the mass-spring system described by the equation \(y'' + 5y' + 4y = 0\) to cross its equilibrium position exactly 5 times.
   (d) (3 points) \(\mathcal{L}\{2f(t) - 5g(t)\} = 2\mathcal{L}\{f(t)\} - 5\mathcal{L}\{g(t)\}\).
   (e) (3 points) \(\mathcal{L}\{u(t - 4) t^2\} = e^{-4s} \frac{2}{s^3}\).
   (f) (3 points) It is possible to separate the partial differential equation, \(4u_{xx} - u_{tt} - 6u = 0\) into two ordinary differential equations.
14. (14 points) In each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = A\mathbf{x}$.

(a) (4 points) Suppose that the system’s general solution is

$$x(t) = C_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-6t}.$$ 

Classify the type and stability of the system’s critical point at $(0,0)$.

(b) (4 points) Suppose the only eigenvalue of the coefficient matrix $A$ is 2, which has corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write down the system’s general solution.

(c) (3 points) Classify the type and stability of the critical point at $(0,0)$ for the system described in (b).

(d) (3 points) Suppose $A$ has eigenvalues $9i$ and $-9i$, classify the type and stability of the system’s critical point at $(0,0)$. 
15. (14 points) Find all positive eigenvalues and corresponding eigenfunctions of the boundary value problem
\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(4) = 0. \]
16. (16 points) Let \( f(x) = x^3, \quad 0 < x < 1. \)

   (a) (4 points) Consider the odd periodic extension, of period \( T = 2 \), of \( f(x) \). Sketch 3 periods, on the interval \(-3 < x < 3\), of this odd periodic extension.

   (b) (2 points) Find \( a_{10} \), the 10th cosine coefficient of the Fourier series of the odd periodic extension in (a).

   (c) (6 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

      \[
      (i) \quad b_n = \frac{1}{2} \int_0^1 x^3 \sin \frac{n\pi x}{2} \, dx \\
      (ii) \quad b_n = \int_{-1}^1 x^3 \sin \frac{n\pi x}{2} \, dx \\
      (iii) \quad b_n = 2 \int_0^1 x^3 \sin(n\pi x) \, dx \\
      (iv) \quad b_n = \int_{-1}^1 x^3 \cos(n\pi x) \, dx
      \]

   (d) (4 points) To what value does the Fourier series converge at \( x = -1 \)? At \( x = \frac{1}{2} \)?
17. (16 points) Suppose the temperature distribution function $u(x, t)$ of a rod that has both ends perfectly insulated is given by the initial-boundary value problem

$$
9u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0
$$

$$
u_x(0, t) = 0, \quad u_x(4, t) = 0,
$$

$$
u(x, 0) = 2 - \cos(\pi x) - 7 \cos(5\pi x).
$$

(a) (14 points) Find the particular solution of the above initial-boundary value problem.

(b) (2 points) What is $\lim_{t \to \infty} u(3, t)$?