MATH 251
Fall 2003
Final Exam
December 15, 2003

NAME:_____________________

ID:____________________

INSTRUCTOR:____________________

There are 14 questions on 13 pages. Please read each problem carefully before starting to solve it. For each multiple choice problem 4 answers are given, only one of which is correct. Mark only one choice. For partial credit questions, all work must be shown - credit will not be given for an answer unsupported by work.

NO CALCULATORS ARE ALLOWED.
PLEASE DO NOT WRITE IN THE BOX BELOW.

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Total:________
1. (6 points) What is the inverse Laplace transform of

\[ F(s) = \frac{4e^{-2s}}{s^2 + 4}. \]

(a) \( u_2(t) \sin(2t - 2) \)
(b) \( 2u_2(t) \sin(2t - 4) \)
(c) \( 4 \sin(2t - 4) \)
(d) \( 2u_3(t) \sin(2t - 2) \)

2. (6 points) Consider the following ordinary differential equation. Choose the wrong statement from the choices below.

\[ \frac{dy}{dt} = y^3 - 6y. \]

(a) Equilibrium solutions are \( y = 0, \sqrt{6}, -\sqrt{6} \).
(b) If \( y(0) = \sqrt{7} \), then \( \lim_{t \to \infty} y(t) = 0 \).
(c) If \( y(0) = \sqrt{6} \), then \( \lim_{t \to \infty} y(t) = 0 \).
(d) 0 is the only stable solution.
3. (6 points) Find the value \( a \) for which the given equation below is exact.

\[
(xy^2 + ax^2 y) + (x + y)x^2 \frac{dy}{dx} = 0
\]

(a) \( a = 1 \)
(b) \( a = 2 \)
(c) \( a = 3 \)
(d) \( a = 4 \)

4. (6 points) Find the correct step-function representation of the following function:

\[
f(t) = \begin{cases} 
  t & t < 1 \\
  e^t + e^{-t} & 1 \leq t < 2 \\
  1 & 2 \geq t 
\end{cases}
\]

(a) \( t + u_1(t)(-t + e^t + e^{-t}) + u_2(t)(1 - e^t - e^{-t}) \)
(b) \( 1 + u_1(t)(-1 + e^t + e^{-t}) + u_2(t)(t - e^t - e^{-t}) \)
(c) \( 1 + u_1(t)(e^t + e^{-t}) - u_2(t)(e^t + e^{-t}) \)
(d) \( u_1(t)(-t + e^t + e^{-t}) + u_2(t)(1 - e^t - e^{-t}) \)
5. (6 points) Consider the Fourier series of the function given below.

\[ f(x) = \begin{cases} 
  x^2 + 1 & -2 < x < \frac{1}{2} \\
  \sin(2\pi x) & \frac{1}{2} \leq x < 2 
\end{cases}, \quad f(x + 4) = f(x) \]

To what value does the Fourier series converge at \( x = \frac{1}{2} \)?

(a) \( \frac{5}{4} \)
(b) \( \frac{5}{8} \)
(c) 0
(d) \( \frac{1}{2} \)
6. (6 points) Which of the following is the general solution to

\[ X' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} X. \]

(a) \[ c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t} \]

(b) \[ c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t \]

(c) \[ c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} e^{5t} \]

(d) \[ c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} e^t \]
7. (a) (14 points) Find the critical points for the system

\[
\frac{dx}{dt} = x^2 - y^2 \\
\frac{dy}{dt} = 1 - x
\]

(b) Linearize the system at any one critical point you like. Specify the stability of the critical point. (stable/asymptotically stable/unstable).
8. (12 points) Solve the following initial value problem.

\[ y'' + 2y' - 3y = t + \delta(t), \quad y(0) = 0 \quad y'(0) = 0 \]
9. (14 points) A mass of 2kg stretches a spring 5m. The system has a damping constant of 6\(\frac{kg}{s}\). The mass is initially at its equilibrium position and set in motion with downward velocity 5\(\frac{m}{s}\). You may take \(g = 10\frac{m}{s^2}\).

(a) Construct and solve the differential equation for this system.

(b) What is the quasi-period of this system?
10. (a) (14 points) Find the critical points for the equation

\[
\frac{dy}{dt} = y(9 - y^2) \quad y(0) = y_0, \quad -\infty < y_0 < \infty
\]

(b) Classify the stability of the critical points as asymptotically stable/unstable.
11. (12 points) Rewrite the following function in terms of step functions and find its Laplace transform.

\[ f(t) = \begin{cases} 
1 & 0 \leq t < \frac{\pi}{2} \\
\sin 2t & \frac{\pi}{2} \leq t 
\end{cases} \]
12. (16 points) Find the Fourier series for the following function

\[ f(x) = \pi^2 - x^2 \quad -\pi \leq x \leq \pi \]

\[ f(x + 2\pi) = f(x) \]
13. (16 points) Find the solution to the Heat Equation with boundary conditions given below:

\[
\begin{align*}
    u_t &= 36u_{xx} \quad 0 < x < 4, \quad t > 0 \\
    u(0, t) &= 0 = u(4, t) \quad t > 0 \\
    u(x, 0) &= \sin \pi x - \sin \frac{5\pi x}{4}
\end{align*}
\]
14. (16 points) Solve the initial value problem

\[ X' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} X \quad X(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \]