This exam has 10 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown.** For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**You may not use a calculator on this exam. Please turn off and put away your cell phone and all other mobile devices.**

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**Total:**
1. (6 points) Find the general solution of the linear equation
\[ y^{(5)} + 2y^{(4)} + 5y''' = 0. \]

2. (5 points) Rewrite the following third order linear equation into an equivalent system of first order linear equations.
\[ y''' + 3y'' - 2y' + 4y = \sin 2t \]
3. (8 points)

(a) (4 points) Evaluate the following definite integral: \[ \int_0^\infty e^{(4-s)t} \cos(3t) \, dt. \]

(Hint: Use the fact that this integral represents the Laplace transform of a certain function. Avoid computing it directly.)

(b) (4 points) Suppose \( \mathcal{L}\{f(t)\} = \frac{2s^2}{s^4 + 100}. \) Use properties of the Laplace transform to determine \( \mathcal{L}\{e^{-\pi t} f(t)\}. \)

4. (5 points) Suppose the linear system \( x' = \begin{bmatrix} 2 - \alpha^2 & 0 \\ 0 & -2\alpha - 1 \end{bmatrix} x \) has an unstable proper node at \((0,0)\). Determine all possible value(s) of \( \alpha. \)
5. (12 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, $x' = Ax$, where $A$ is a 2x2 matrix of real numbers. Based solely on the information given in each part, determine the type and stability of the system’s critical point at (0, 0).

(a) Eigenvalues of $A$ are $-1$ and $-6$.

(b) Eigenvalues of $A$ are $3 + 7i$ and $3 - 7i$.

(c) Eigenvalues of $A$ are $9i$ and $-9i$.

(d) Eigenvalues of $A$ are $\sqrt{11}$ and $\pi^2$.

(e) The general solution is $x(t) = C_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(f) The general solution is $x(t) = C_1 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -1 \\ \sqrt{3} + t \end{bmatrix}$.
6. (14 points) Find the inverse Laplace transform of each function given below.

(a) (7 points) \[ F(s) = \frac{s^2 + 11s - 31}{(s + 5)(s^2 + 36)} \]

(b) (7 points) \[ F(s) = e^{-s} \frac{-5s + 2}{s^2 + 4s + 20} \]
7. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step function.

$$
f(t) = \begin{cases} 
2t^2 - e^{-6t}, & 0 \leq t < 4 \\
9t + 3, & 4 \leq t 
\end{cases}
$$

Then find its Laplace transform.
8. (14 points) Consider the initial value problem
\[ y'' + 4y = \delta(t - \pi) + u_{10}(t), \quad y(0) = 1, \quad y'(0) = -4. \]

(a) (12 points) Use the Laplace transform to solve the initial value problem.

(b) (1 point) Evaluate \( y\left(\frac{\pi}{2}\right) \).

(c) (1 point) Evaluate \( y(2\pi) \).
9. (12 points)

(a) (10 points) Solve the initial value problem

\[
\mathbf{x}' = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.
\]

(b) (2 points) Classify the type and stability of the critical point at (0, 0).
10. (12 points) Consider the nonlinear system:

\[ x' = (x + 1)(y - 2) = xy - 2x + y - 2 \]
\[ y' = (x + y)(2x - y) = 2x^2 + xy - y^2 \]

(a) (4 points) The system has 4 critical points. One of the critical points of the system is \((-1, 1)\). Find the other 3 critical points of the system.

(b) (4 points) Linearize the system about the critical point \((-1, 1)\). Identify the coefficient matrix of the linearized system.

(c) (4 points) What are the eigenvalues of the coefficient matrix? Classify the type and stability of the critical point at \((-1, 1)\) by examining the linearized system found in (b).