This exam has 12 questions for a total of 100 points. In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

Please turn off and put away your cell phone.

You may not use a calculator on this exam.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
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<tr>
<td>1 through 7</td>
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<td>8</td>
<td>(12)</td>
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<td>9</td>
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<td>11</td>
<td>(15)</td>
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<tr>
<td>12</td>
<td>(11)</td>
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Total:
1. (5 points) Evaluate the following definite integral
\[ \int_{0}^{\infty} e^{-(s-3)t} \cos(2t) \, dt. \]

(a) \[ \frac{s - 3}{(s - 3)^2 + 4} \]

(b) \[ \frac{s}{(s - 3)(s^2 + 4)} \]

(c) \[ \frac{s}{(s + 3)(s^2 + 4)} \]

(d) \[ \frac{s + 3}{(s + 3)^2 + 4} \]

2. (5 points) Find the Laplace transform \( \mathcal{L}\{u_3(t)(9 - t^2)\} \).

(a) \[ F(s) = e^{-3s} \frac{9s^2 - 2}{s^3} \]

(b) \[ F(s) = e^{-3s} \frac{6s - 2}{s^3} \]

(c) \[ F(s) = e^{-3s} \frac{-6s - 2}{s^3} \]

(d) \[ F(s) = e^{-3s} \frac{9s^2 - 2}{s^4} \]
3. (5 points) Find the inverse Laplace transform of \( F(s) = \frac{s - 3}{s^2 + 2s + 5} \).

(a) \( f(t) = e^{-t} \cos(2t) - 3e^{-t} \sin(2t) \)
(b) \( f(t) = e^t \cos(2t) - e^t \sin(2t) \)
(c) \( f(t) = e^t \cos(2t) - 3e^t \sin(2t) \)
(d) \( f(t) = e^{-t} \cos(2t) - 2e^{-t} \sin(2t) \)

4. (5 points) Let \( \mathbf{x}(t) \) be the solution of the initial value problem

\[
\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ -3 & -4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} \alpha \\ 6 \end{bmatrix}.
\]

Suppose \( \lim_{t \to \infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), find the value of \( \alpha \).

(a) \( \alpha = -2 \)
(b) \( \alpha = 3 \)
(c) \( \alpha = -12 \)
(d) \( \alpha = 18 \)
5. (5 points) Consider a certain $2 \times 2$ linear system $\mathbf{x}' = A\mathbf{x}$, where $A$ is a matrix of real numbers. Suppose some of its solutions do not reach a limit either as $t \to +\infty$, or as $t \to -\infty$. Then the critical point $(0, 0)$ must be

(a) a saddle point.
(b) a center.
(c) either a saddle point or a center.
(d) neither a saddle point nor a center.

6. (5 points) How many critical points does the following system have?

\[
\begin{align*}
x' &= 2 + xy \\
y' &= x + 2y
\end{align*}
\]

(a) 2
(b) 3
(c) 4
(d) More than 4.
7. (5 points) Given that (2, −1) is a critical point of the system

\[
\begin{align*}
x' &= 2 + xy \\
y' &= x + 2y
\end{align*}
\]

Which of the following statements is TRUE regarding (2, −1)?

(a) It is an asymptotically stable spiral point.
(b) It is an unstable spiral point.
(c) It is an asymptotically stable node.
(d) It is an unstable saddle point.
8. (12 points) For each part below, determine whether the statement is true or false. To receive credit you must (briefly) justify each answer.

(a) \( \mathcal{L}\{e^{-(2+3t)}(\cos(t) - \sin(t))\} = \frac{1}{e^2} \mathcal{L}\{e^{-3t}\cos(t)\} - \frac{1}{e^2} \mathcal{L}\{e^{-3t}\sin(t)\} \).

(b) Suppose \( f(t) = u_3(t) \cos t + u_2(t)t + u_4(t) \), then \( f(\pi) = \pi \).

(c) Suppose \( f(3) = e \), then \( \mathcal{L}\{\delta(t - 3) \ln(f(t))\} = e^{-3s} \).

(d) The function

\[
\begin{align*}
  f(t) &= \begin{cases} 
    2, & \text{if } t < \pi \\
    t^2, & \text{if } \pi \leq t < 3\pi \\
    t, & \text{if } 3\pi \leq t 
  \end{cases}
\end{align*}
\]

can be rewritten as \( f(t) = 2 + u_\pi(t)(t^2 - 2) + u_{3\pi}(t)(t - t^2) \).
9. (12 points) Consider the systems of linear differential equations listed below.

\[ A. \quad x' = \begin{bmatrix} -5 & 2 \\ 0 & -5 \end{bmatrix} x \]
\[ B. \quad x' = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix} x \]
\[ C. \quad x' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} x \]
\[ D. \quad x' = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} x \]

For each of parts (a) through (f) below, write down the letter corresponding to the system on this list that has the indicated property. There is only one correct answer to each part. However, a system may be re-used for more than one part.

(a) Which system is (neutrally) stable?

(b) Which system has a proper node at (0, 0)?

(c) Which system has a saddle point at (0, 0)?

(d) Which system has all of its solutions converge to (0, 0) as \( t \to \infty \)?

(e) This system’s coefficient matrix has only one linearly independent eigenvector.

(f) Every nonzero vector is an eigenvector of this system’s coefficient matrix.
10. (15 points)

(a) (5 points) Suppose \( L\{f(t)\} = \frac{s}{s^3 + 11} \). What is \( L\{e^{-2t} f(t)\} \)?

(b) (5 points) Rewrite the following third order linear equation into an equivalent system of first order linear equations.

\[
y''' + e^{5t}y'' + 3y' - \sin(2t)y = t^3.
\]

(c) (5 points) Find all values of \( \beta \) for which the critical point \((0, 0)\) of the linear system below is a (neutrally) stable center.

\[
x' = \begin{bmatrix} \beta & -9 \\ 4 & -\beta \end{bmatrix} x
\]
11. (15 points) Use the Laplace transform to solve the following initial value problem.

\[ y'' + 4y' + 3y = \delta(t - 1) + 2u_6(t)(t - 6), \quad y(0) = 1, \quad y'(0) = 0. \]
12. (11 points) Consider the initial value problem.

\[ x' = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} x, \quad x(e^2) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}. \]

(a) (9 points) Solve the initial value problem.

(b) (2 points) Classify the type and stability of the critical point at \((0,0)\).
<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
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<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>$t^n$, $n = \text{positive integer}$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
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<tr>
<td>$p^p$, $p &gt; -1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}$</td>
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<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
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<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
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<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
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<tr>
<td>$\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
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<tr>
<td>$e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$</td>
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<tr>
<td>$e^{at} \cos bt$</td>
<td>$\frac{s - a}{(s-a)^2 + b^2}$</td>
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<tr>
<td>$t^n e^{at}$, $n = \text{positive integer}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$</td>
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<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
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<td>$u_c(t)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
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<tr>
<td>$e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
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<td>$f(ct)$</td>
<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$</td>
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<tr>
<td>$(f * g)(t) = \int_0^t f(t-\tau)g(\tau),d\tau$</td>
<td>$F(s)G(s)$</td>
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<tr>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
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<tr>
<td>$f^{(n)}(t)$</td>
<td>$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$</td>
</tr>
<tr>
<td>$(-t)^n f(t)$</td>
<td>$F^{(n)}(s)$</td>
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