This exam has 11 questions for a total of 100 points. In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for answers not supported by work.

You may not use a calculator on this exam. Please turn off and put away your cell phone.

The last sheet of the booklet contains a table of Laplace transforms and may be detached.
1. (5 points) Match the phase portraits with their names by entering the numbers in the blanks below.

(a) node
(b) saddle
(c) spiral
(d) center
(e) proper node

2. (5 points) Consider the mechanical system represented by the ODE:

\[ y'' + 4y' + 5y = 0. \]

The system is

(a) underdamped
(b) critically damped
(c) overdamped
(d) in resonance
3. (5 points) Let \( f(t) = u(t-2)e^{t-2}\sin(3t-6) \), where

\[
\begin{align*}
    u(t-c) = u_c(t) &= \begin{cases} 
        0 & \text{if } t < c \\ 
        1 & \text{if } c \leq t 
    \end{cases}
\end{align*}
\]

The Laplace transform of \( f(t) \) is

(a) \( e^{-2s} \frac{1}{s-1} \frac{3}{s^2+9} \)
(b) \( e^{-2s} \frac{3}{(s-1)^2+9} \)
(c) \( e^{-2s} \frac{1}{s-1} \frac{s}{s^2+9} \)
(d) \( e^{2s} \frac{s-1}{(s-1)^2+9} \)

4. (5 points) Given a matrix \( A = \begin{pmatrix} 4 & -6 \\ 0 & -2 \end{pmatrix} \). Then the origin \( (0,0) \) of \( x' = Ax \) is

(a) asymptotically stable spiral
(b) unstable saddle point
(c) stable center
(d) unstable spiral point
5. (5 points) Evaluate
\[ \int_{0}^{\infty} e^{-st} t \, dt, \quad (s > 0) \]

(a) \(-s\)
(b) \(s^{-1}\)
(c) \(e^{-s}\)
(d) \(s^{-2}\)

6. (5 points) Match the five names for the critical point 0 of a homogeneous linear system of ODE’s \( x' = Ax \) with the five general solutions given in the table below by placing one of the letters \( N, S, P, C, \) or \( L \) in each of the five blanks. (Each letter should be used exactly once.)

| (i) \( c_1 e^t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) | \( N \) Node |
| (ii) \( c_1 e^t \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix} \) | \( S \) Saddle |
| (iii) \( c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) | \( P \) Proper node |
| (iv) \( c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) | \( C \) Center |
| (v) \( c_1 \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix} \) | \( L \) spiraL |
7. Consider the spring-mass system with damping and external force that is modeled by the following ODE:

\[ y'' + \gamma y' + 4y = F_0 \cos(\omega t). \]

(a) (8 points) If \( \gamma = 0 \) and \( F_0 = 0 \), then what is the amplitude of the displacement \( y \) of the system with \( y(0) = 1 \), \( y'(0) = -2 \)?

(b) (4 points) If \( F_0 = 0 \), then what is the smallest value of \( \gamma \) for which the object crosses its equilibrium position a finite number of times.

(c) (4 points) If \( F_0 = 3 \), then find a value for \( \gamma \) and a value for \( \omega \) so that the system is in resonance.
8. In the following parts determine the form of a particular solution with as few constants as possible. Do **NOT** solve for the undetermined coefficients!

(a) (4 points) \( y'' + y' = t^2 + e^{-t} \)

(b) (4 points) \( y'' + 2y' + y = te^{-t} \)

(c) (4 points) \( y'' + 2y' + y = t^2 e^{2t} \cos(4t) \)
9. (a) (6 points) Express the following function by means of a single formula using unit-step functions $u(t - c)$, with various choices of $c$:

$$f(t) = \begin{cases} 
  t & \text{for } 0 \leq t < 2, \\
  2 & \text{for } 2 \leq t < 4, \\
  0 & \text{for } 4 \leq t 
\end{cases}$$

(b) (6 points) Compute the Laplace transform of

$$g(t) = t^2 + u(t - 3)e^t - u(t - \frac{\pi}{2}) \cos(t)$$
10. (a) (5 points) Assume that the acceleration due to gravity is \( g = 10 \text{ meters/sec}^2 \).
   An object whose mass is 2 kg stretches a spring 4 meters to equilibrium.
   At time \( t = 0 \) the object is released 1 meter above its equilibrium position with a downward velocity of 2 meters/sec. At time \( t = 3 \) a constant force of 5 Newtons in the upward direction is applied. Write an ODE and initial conditions that represent this spring-mass system. **DO NOT SOLVE IT.**

(b) (10 points) Solve using Laplace transform:

\[
y'' + 2y' + 2y = \delta(t - 5), \quad y(0) = 1, \quad y'(0) = 2,
\]

where \( \delta(t) \) is the Dirac delta function.
11. (15 points) Solve the following linear homogeneous system of first order ODE’s:

\[ x' = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \]
<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
<th>Inverse Laplace Transform</th>
</tr>
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<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
<td>$s &gt; a$</td>
</tr>
<tr>
<td>$t^n$, $n$ positive integer</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$s &gt; 0$</td>
</tr>
<tr>
<td>$t^p$, $p &gt; -1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}$</td>
<td>$s &gt; 0$</td>
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<td>$\cos at$</td>
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<td>$\cosh at$</td>
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<tr>
<td>$e^{at}\sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$</td>
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<tr>
<td>$e^{at}\cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$</td>
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<tr>
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<td>$\frac{n!}{(s-a)^{n+1}}$</td>
<td>$s &gt; a$</td>
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<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
<td>$s &gt; 0$</td>
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<td>$e^{-cs}F(s)$</td>
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<td>$e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
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<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$</td>
<td>$c &gt; 0$</td>
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<td>$F(s)G(s)$</td>
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<td>$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$</td>
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<td>$(-t)^\alpha f(t)$</td>
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