This exam has 13 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number.

**You may not use a calculator on this exam. Please turn off and put away your cell phone.**

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1. (5 points) Which of the following equations is a second order linear ordinary differential equation?

(a) \((1 + y^2)y'' + ty' = e^t\)
(b) \(e^t y'' + t^2 y = \ln t\)
(c) \(y' + ty^2 = 0\)
(d) \(y' + (\sin t)y = 0\)

2. (5 points) Determine the differential equation whose direction field is given below.

(a) \(y' = y - x\)
(b) \(y' = x + y\)
(c) \(y' = y(y - 2)\)
(d) \(y' = x(x - 2)\)
3. (5 points) Which of the following is an integrating factor for the differential equation

\[ ty' - (t + 2)y = 2 \tan(3t) \]?

DO NOT solve this differential equation.

(a) \( \mu(t) = e^{-t}t^{-2} \)

(b) \( \mu(t) = \frac{t + 2}{t} \)

(c) \( \mu(t) = \frac{1}{t^2} \)

(d) \( \mu(t) = t^2 e^t \)

4. (5 points) Consider the initial value problem

\[ (t^2 - 4)y' + 3y = \ln |5 - t|, \quad y(3) = 0. \]

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(a) \((2, \infty)\)

(b) \((-2, 2)\)

(c) \((2, 5)\)

(d) \((-\infty, 5)\)
5. (5 points) Consider the equation:
\[
(\beta x^2 y + xy^2 + \arctan x) + (x^3 + x^2 y + \cot y)y' = 0.
\]
Find the value of $\beta$ such that the above equation is exact.

(a) $\beta = -1$
(b) $\beta = 1$
(c) $\beta = -3$
(d) $\beta = 3$

6. (5 points) Suppose $y_1(t)$ and $y_2(t)$ are any two solutions of the second order linear equation
\[
y'' + 2 \cot(t) y' + ty = 0.
\]
What is the general form of their Wronskian, $W(y_1, y_2)(t)$?

(a) $C \csc^2 t$
(b) $C \sin^2 t$
(c) $C \sec^2 t$
(d) $C \cos^2 t$
7. (5 points) \(y_1(t) = te^t\) and \(y_2(t) = e^t\cos 2t\) are both solutions of the second order linear equation
\[y'' + p(t) y' + q(t) y = 0\]
Which statement below is TRUE?

(a) \(y = (10t - 5\cos 2t)e^t\) is not a solution of the equation.
(b) \(y = 0\) can never be a solution of the equation.
(c) \(y_1\) and \(y_2\) is a fundamental pair of solutions
(d) \(W(y_1, y_2)(t) = 0\)

8. (5 points) Consider the fourth order linear equation
\[y^{(4)} + 4y^{(3)} + 3y'' = 0\]
What is its general solution?

(a) \(y(t) = C_1 + C_2 t + C_3 e^{-t} + C_4 e^{-3t}\)
(b) \(y(t) = C_1 + C_2 e^{-t} + C_3 e^{-3t}\)
(c) \(y(t) = C_1 e^t + C_2 e^{3t}\)
(d) \(y(t) = C_1 + C_2 t + C_3 e^t + C_4 e^{3t}\)
9. (14 points) Consider the equation

\[ 2xy^3 + 3x^2y^2y' = 0. \]

(a) (3 points) Is it a separable equation? Why or why not?

(b) (3 points) Verify that the equation is an exact equation.

(c) (6 points) Find its general solution. You may leave your answer in implicit form.

(d) (2 points) Find the solution satisfying the initial condition \( y(1) = 1 \).
10. (10 points) An object with mass \( m = 0.5 \) kg is thrown upward with initial velocity \( v_0 = 20 \) meters per second from the roof of a building which is 30 meters high. Assume there is a force due to air resistance that is proportional to the velocity \( v \) of the object with a positive constant of proportionality (coefficient of drag) \( k = \frac{1}{10} \). You may use \( g = 10 \) meters per second squared as the gravitational constant.

(a) (4 points) Construct an initial value problem that models for velocity of the object in motion at any time \( t \).

(b) (4 points) Solve the initial value problem to find \( v(t) \).

(c) (2 points) Suppose the motion could continue indefinitely. To what value would the object’s velocity approach eventually?
11. (13 points) Consider the autonomous differential equation
\[ y' = -y^4 + 16y^2. \]

(a) (3 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) If \( y(t) \) is a solution that satisfies \( y(-1) = -4 \), what is \( y(0) \)? Without solving the equation, briefly explain your conclusion.

(d) (2 points) If \( y(t) \) is a solution that satisfies \( y(3) = -3 \), then what is \( \lim_{t \to \infty} y(t) \)?
12. (13 points) Consider the second order nonhomogeneous linear equation
\[ y'' + 4y' = 2e^t + t. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (7 points) Find the general solution of the equation.

(c) (3 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.

\[ y'' + 4y' = 11te^{-4t}(\sin 6t - 3 \cos 6t) + 7e^{-4t}. \]
13. (10 points) A mass-spring system is described by the equation

\[ 5u'' + \gamma u' + ku = F(t). \]

(a) (2 points) Suppose the mass originally stretched the spring 2 meters to reach its equilibrium position. What is the spring constant \( k \)? (Assume \( g = 10 \text{ m/s}^2 \) to be the gravitational constant.)

(b) (2 points) Suppose \( k = 20 \). For what value(s) of \( \gamma \) would this system be critically damped?

(c) (2 points) Suppose \( \gamma = 0 \) and \( k = 45 \). What is the natural frequency of this system?

(d) (2 points) True or false: Suppose \( \gamma = 0, k = 5 \), and \( F(t) = 7 \sin t \), then the mass-spring system is undergoing resonance?

(e) (2 points) Suppose \( \gamma = 3, k = 1, F(t) = 0 \). What is the quasi-frequency the system?