This exam has 14 questions for a total of 100 points. Show all your work! In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.

Do not write in this box.

1
through
10:________ (50)
11:________ (12)
12:________ (10)
13:________ (15)
14:________ (13)
Total:________
1. (5 points) Which of the equations below is a nonlinear second order differential equation?

   (a) \((y')^2 + \cos(2t) = 0\)
   
   (b) \((y'')^2 + y' + e^t = 0\)
   
   (c) \(y'' + t^3y' + e^t = y\)
   
   (d) \(y'' + \cos(2t)y' + e^t y = 0\)

2. (5 points) Consider the initial value problem

   \((t^2 - 1)y' + \ln(t)y = \tan(2\pi t), \quad y(0.875) = 10.\)

   According to the existence and uniqueness theorem, what is the largest interval in which a unique solution is guaranteed to exist?

   (a) \((0, \frac{1}{4})\)
   
   (b) \((\frac{1}{4}, \frac{3}{4})\)
   
   (c) \((\frac{3}{4}, 1)\)
   
   (d) \((\frac{1}{4}, 1)\)
3. (5 points) Which statement below is FALSE?

(a) Every exact equation is separable.
(b) Every autonomous equation is separable.
(c) A first order linear equation might also be separable.
(d) A first order linear equation might not be separable.

4. (5 points) What is a suitable integrating factor for solving the following linear equation

\[(1 - t^2)y' + 4y + t^2 = 0, \quad -1 < t < 1?\]

(a) \(\mu(t) = (\ln(1 - t^2))^{-2}\)
(b) \(\mu(t) = \left(\frac{1 + t}{1 - t}\right)^2\)
(c) \(\mu(t) = e^{4t}\)
(d) \(\mu(t) = e^{4\text{arctan}(t)}\)
5. (5 points) An aircraft weighing 16000 kg produces a constant propulsive force of 121000 kg.meter/second$^2$. Assume that the drag force (air resistance) acting on this aircraft is directly proportional to the square of the velocity and the drag coefficient is 10. Let $v(t)$ be the velocity of the aircraft as a function of time $t$. Assume the aircraft is travelling at a velocity of 9 meter/second at $t = 5$ second. Neglecting any other forces that may be acting on the aircraft, determine which of the following initial value problems is the best model for the velocity of this aircraft.

(a) $16000v' = 121000 - 10v^2, \quad v(0) = 9$
(b) $16000v' = 121000 + 10v^2, \quad v(0) = 9$
(c) $16000v' = 121000 + 10v^2, \quad v(5) = 9$
(d) $16000v' = 121000 - 10v^2, \quad v(5) = 9$

6. (5 points) Consider the following first order differential equation

$$e^x \sin(y) - 2y \sin(x) + (e^x \cos(y) + 2 \cos(x))y' = 0.$$ 

Which of the following families of functions satisfies the above equation?

(a) $e^x \cos(y) + y^2 \sin(x) - C = 0$
(b) $e^x \cos(y) - y^2 \cos(x) - C = 0$
(c) $e^x \sin(y) - 2y \cos(x) - C = 0$
(d) $e^x \sin(y) + 2y \cos(x) - C = 0$
7. (5 points) Suppose \( y_1(t) \) and \( y_2(t) \) are any two solutions of the second order linear equation
\[
y'' + 4 \tan(t)y' + t \cos(t)y = 0.
\]
What is the general form of their Wronskian \( W(y_1, y_2)(t) \)?

(a) \( W(y_1, y_2)(t) = C \sin^4 t \)
(b) \( W(y_1, y_2)(t) = C \sec^4 t \)
(c) \( W(y_1, y_2)(t) = C \cos^4 t \)
(d) \( W(y_1, y_2)(t) = Ct^4 \ln t \).

8. (5 points) Suppose \( y_1(t) = 3te^{-t} \) and \( y_2(t) = (1 + t^2)e^{-t} \) are both solutions of the second order linear equation
\[
y'' + p(t)y' + q(t)y = 0.
\]
Which one of statements (a) through (c) below is FALSE?

(a) \( y(t) = (1 + 6t + t^2)e^{-t} \) is also a solution.
(b) \( y(t) = 0 \) is also a solution.
(c) \( y_1 \) and \( y_2 \) form a set of fundamental solutions.
(d) All 3 statements are true.
9. (5 points) Which equation below has the property that all of its solutions tend to zero as $t \to \infty$?

   (a) $y'' + 6y' + 9y = 0$
   (b) $y'' - 4y' - 5y = 0$
   (c) $y'' - 4y' + 5y = 0$
   (d) $y'' + 25y = 0$

10. (5 points) Consider the fourth order linear equation

$$y^{(4)} - 10y'' + 25y = 0.$$ 

Which of the following is the general solution?

   (a) $(C_1 + C_2t) \cos \sqrt{5}t + (C_3 + C_4t) \sin \sqrt{5}t$
   (b) $C_1e^{-\sqrt{5}t} + C_2te^{-\sqrt{5}t} + C_3e^{\sqrt{5}t} + C_4te^{\sqrt{5}t}$
   (c) $C_1e^{-\sqrt{5}t} + C_2te^{-\sqrt{5}t} + C_3t^2e^{-\sqrt{5}t} + C_4t^3e^{-\sqrt{5}t}$
   (d) $C_1e^{-\sqrt{5}t} + C_2e^{\sqrt{5}t} + C_3 \cos \sqrt{5}t + C_4 \sin \sqrt{5}t$
11. (12 points) Consider various mass-spring systems and the differential equations that describe their displacement. A list of equations is given below. Each equation may or may not describe the displacement of any mass-spring system.

A. \( y'' + 2y' + 5y = 12 \)
B. \( 2y'' + 4y = 9 \sin 2t \)
C. \( 4y'' + 4y = 0 \)
D. \( y'' + 5y' + 4y = 0 \)
E. \( y'' - 6y' + 9y = 0 \)
F. \( 2y'' + 18y = 5 \cos 3t \)
G. \( 3y'' + 6y' + 3y = 0 \)

For each of parts (a) through (f) below, write down the letter corresponding to the equation on the list above describing the correct mass-spring system with the specified behavior. There is only one correct equation to each part. However, an equation may be re-used for more than one part.

(a) (2 points) This system is critically damped.

(b) (2 points) This system is overdamped.

(c) (2 points) This system is undergoing resonance.

(d) (2 points) This system has a natural period of \( 2\pi \) seconds.

(e) (2 points) This system’s displacement is always simple harmonic.

(f) (2 points) As \( t \to \infty \), this system’s displacement approaches a nonzero limit.
12. (10 points) Solve the following initial value problem:

\[ y' = \frac{3 \cos(3x)}{y + 2}, \quad y(0) = -5. \]

Simplify your answer into an **explicit** form.
13. (15 points) Consider the autonomous differential equation

\[ y' = 36y^2 - 9y^4. \]

(a) (3 points) Find all of its equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

(c) (2 points) Suppose \(y_1(t)\) and \(y_2(t)\) are two solutions of the equation such that \(y_1(17) = -1\) and \(y_2(29) = 1\). What is \(\lim_{t \to \infty} (y_1(t) + y_2(t))\)?

(d) (2 points) Suppose \(y(-1729) = \beta\), and that \(\lim_{t \to \infty} y(t) = 2\). Find all possible values of \(\beta\).

(e) (2 points) If \(y(-53) = -2\), then what is \(y(42)\)? Without solving the equation, briefly explain your conclusion.
14. (13 points) Consider the second order nonhomogeneous linear equation
\[ y'' + 2y' + 2y = 5e^t + 2t + 2. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (7 points) Find its general solution.

(c) (3 points) What is the **form** of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**
\[ y'' + 2y' + 2y = 2te^{-t} \sin t. \]