There are 8 multiple choice questions and 6 partial credit questions. In order to obtain full credit for the partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work on a partial credit problem. THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.

For multiple choice problems, write the letter of your choice in the space provided below.

<table>
<thead>
<tr>
<th>Your Answer</th>
<th>Points awarded</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (5 pts)</td>
<td>Q. 9 (15 pts)</td>
</tr>
<tr>
<td>2. (5 pts)</td>
<td>Q. 10 (15 pts)</td>
</tr>
<tr>
<td>3. (5 pts)</td>
<td>Q. 11 (20 pts)</td>
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<td>4. (5 pts)</td>
<td>Q. 12 (20 pts)</td>
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<tr>
<td>5. (5 pts)</td>
<td>Q. 13 (20 pts)</td>
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<tr>
<td>6. (5 pts)</td>
<td>Q. 14 (20 pts)</td>
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<tr>
<td>7. (5 pts)</td>
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<tr>
<td>8. (5 pts)</td>
<td></td>
</tr>
</tbody>
</table>
1. (5 points) Find two linearly independent vectors
   
   (a) \( \left( \begin{array}{c} 1 \\ 2 \end{array} \right), \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \). 
   
   (b) \( \left( \begin{array}{c} 1 \\ -1 \end{array} \right), \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \). 
   
   (c) \( \left( \begin{array}{c} 1 \\ 2 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \). 
   
   (d) None of the above.

2. (5 points) Let \( y_1(t) = \sin t \) and \( y_2(t) = \cos t \). Which of the following is not true?
   
   (a) \( y_1 \) and \( y_2 \) are linearly independent.
   
   (b) \( y_1 \) and \( y_2 \) are solutions of \( y'' + y = 0 \).
   
   (c) Wronskian \( W(y_1, y_2) = 0 \).
   
   (d) \( e^{it} = y_2 + iy_1 \).

3. (5 points) The function \( y_1(t) = u_2(t)(0.5 - e^{-(t-2)} + 0.5e^{2(t-2)}) \) is the solution of \( y'' + 3y' + 2y = u_2(t) \), \( y(0) = 0 \), \( y'(0) = 0 \), where \( u_2 \) is a Heaviside function. Find the solution of this equation with initial conditions \( y(0) = 0 \), \( y'(0) = 1 \) (see Problem 6(6.4)).
   
   (a) \( y(t) = e^{-t} + e^{-2t} + u_2(t)(0.5 - e^{-(t-2)} + 0.5e^{2(t-2)}) \).
   
   (b) \( y(t) = -e^t + e^{2t} + u_2(t)(0.5 - e^{-(t-2)} + 0.5e^{2(t-2)}) \).
   
   (c) \( y(t) = e^{-t} - e^{-2t} + u_2(t)(0.5 - e^{-(t-2)} + 0.5e^{2(t-2)}) \).
   
   (d) \( y(t) = 0.5 - e^{-(t-2)} + 0.5e^{2(t-2)} \).

4. (5 points) Suppose \( y_1(t) \) is a solution of \( y' + p(t)y = 0 \) and \( y_2(t) \) is a solution of \( y' + p(t)y = g(t) \). Which if the following is not true? (see Problem 25(2.4)).
   
   (a) \( y_2(t) + y_1(t) \) is a solution of \( y' + p(t)y = g(t) \).
   
   (b) \( y_2(t) - y_1(t) \) is a solution of \( y' + p(t)y = g(t) \).
   
   (c) \( -y_2(t) + y_1(t) \) is a solution of \( y' + p(t)y = g(t) \).
   
   (d) \( 2y_2(t) \) is a solution of \( y' + p(t)y = 2g(t) \).
5. (5 points) Suppose
\[ f(t) = \begin{cases} 
  t/2, & \text{if } 0 \leq t < 6, \\
  3, & \text{if } 6 \leq t.
\end{cases} 
\]
Which of the following is not true? (see problem 9(6.4))
(a) \( y = 0.5(\sin t + t) - 0.5u_6(t - 6 - \sin(t - 6)) \) is the solution of the initial value problem \( y'' + y = f(t), \ y(0) = 0, \ y'(0) = 1. \)
(b) The Laplace transform of \( f(t) \) is given by the following expression \( 3 \int_0^\infty e^{-st}dt + \int_0^6(t/2 - 3)e^{-st}dt. \)
(c) \( f(t) \) is a piece-wise continuous function.
(d) \( \int_0^\infty f(t)dt \) converges.

6. (5 points) Problem 16(2.1). Suppose \( y(t) \) is the solution of the initial value problem
\[ \frac{dy}{dt} + \left(\frac{2}{t}\right)y = \frac{(\cos t)}{t^2}, \ y(\pi) = 0, \ t > 0. \] Find \( y(\pi/2). \)
(a) \( 4/\pi^2. \)
(b) \( 0. \)
(c) \( 1. \)
(d) None of the above.

7. (5 points) Problem 10(2.5). For an autonomous system \( y' = y(1-y^2), \) determine stability of its equilibrium solutions.
(a) \( y = -1 \) and \( y = 1 \) are stable, \( y = 0 \) is unstable.
(b) \( y = -1 \) and \( y = 1 \) are unstable, \( y = 0 \) is stable.
(c) \( y = -1 \) and \( y = 0 \) are stable, \( y = 1 \) is unstable.
(d) \( y = -1 \) and \( y = 0 \) are unstable, \( y = 1 \) is stable.

8. (5 points) Problem 21(2.4) Consider the initial value problem \( y' = y^{1/3}, \ y(0) = 0. \) Which of the following is not true.
(a) \( y = 0, \ y = \left(\frac{2}{3}t\right)^{3/2} \) and \( y = -\left(\frac{2}{3}t\right)^{3/2} \) are solutions of this initial value problem
(b) This initial value problem has only three different solutions.
(c) \( y = u_1(t)(\frac{2}{3}(t - 1))^{3/2} \) is a solution of this initial value problem, where \( u_1(t) \) is a Heaviside function.
(d) Nonuniqueness of the solutions of this initial value problem does not contradict the existence and uniqueness theorem.
9. (15 points) Problem 16(2.4).
   (a) Solve the initial value problem \( y' = \frac{t^2}{y(t^3+1)} \), \( y(0) = y_0 \).

   (b) Determine how the interval in which the solution exists depends on the initial value \( y_0 \).
10. (15 points) Problem 15(3.6). Solve the initial value problem \( y'' - 2y' + y = te^t + 4, \ y(0) = 1, \ y'(0) = 1. \)
11. (20 points)

(a) Problem 7(5.6). Solve the initial value problem \(y'' + y = \delta(t - 2\pi)\cos t, \quad y(0) = 0, \quad y'(0) = 1.\)

(b) Write the solution \(y(t)\) as a piecewise defined function (using curly braces instead of Heaviside functions).

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12. (20 points)

(a) Find the general solution to the system $X'(t) = AX$, where $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

(b) Describe the behavior of solutions of the system in part a) as $t$ approaches negative infinity.

(c) Solve the initial value problem for the system in part a) when $X = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
   For answer, see problem 9(7.1).
13. (20 points) Find eigenvalues and classify the type of critical point at the origin of the linear system \( X'(t) = AX \) if the \( 2 \times 2 \) matrix \( A \) is given as

\[
\begin{array}{c|c|c|c}
A & \text{Problem \# in textbook} & \text{Eigenvalues} & \text{Type of Critical Point} & \text{Stability} \\
\hline
(a) \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} & 2(7.5) \\
(b) \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} & 1(7.5) \\
(c) \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} & 6(7.6) \\
(d) \begin{pmatrix} 3/4 & -2 \\ 1 & -5/4 \end{pmatrix} & 11(7.6) \\
(e) \begin{pmatrix} -4/5 & 2 \\ -1 & 6/5 \end{pmatrix} & 12(7.6) \\
(f) \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} & 1(7.8) \\
\end{array}
\]

(g) Draw the phase portraits for the matrices above.
14. (20 points)
   (a) Problem 13, Section 9.3. Find all equilibrium points of \( \frac{dx}{dt} = x - y^2, \frac{dy}{dt} = y - x^2 \).

   (b) Determine stability of these equilibrium points.
### Laplace Transform table

<table>
<thead>
<tr>
<th>( f(t) = \mathcal{L}^{-1}{F(s)} )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( 1 )</td>
<td>( \frac{1}{s}, \quad s &gt; a )</td>
</tr>
<tr>
<td>2. ( e^{at} )</td>
<td>( \frac{1}{s-a}, \quad s &gt; a )</td>
</tr>
<tr>
<td>3. ( te^{at} )</td>
<td>( \frac{1}{(s-a)^2}, \quad s &gt; a )</td>
</tr>
<tr>
<td>4. ( t^2e^{at} )</td>
<td>( \frac{2}{(s-a)^3}, \quad s &gt; a )</td>
</tr>
<tr>
<td>5. ( t^n e^{at} )</td>
<td>( \frac{n!}{(s-a)^{n+1}}, \quad s &gt; a )</td>
</tr>
<tr>
<td>6. ( e^{at} \sin bt )</td>
<td>( \frac{b}{(s-a)^2 + b^2}, \quad s &gt; a )</td>
</tr>
<tr>
<td>7. ( e^{at} \cos bt )</td>
<td>( \frac{s-a}{(s-a)^2 + b^2}, \quad s &gt; a )</td>
</tr>
<tr>
<td>8. ( u_c(t) f(t - c) )</td>
<td>( e^{-cs} F(s) )</td>
</tr>
<tr>
<td>9. ( \delta(t - c) )</td>
<td>( e^{-cs} )</td>
</tr>
<tr>
<td>10. ( f(t) \delta(t - c) )</td>
<td>( e^{-cs} f(c) )</td>
</tr>
<tr>
<td>11. ( f'(t) )</td>
<td>( sF(s) - f(0) )</td>
</tr>
<tr>
<td>12. ( f''(t) )</td>
<td>( s^2F(s) - sf(0) - f'(0) )</td>
</tr>
</tbody>
</table>