There are 6 multiple choice questions and 6 partial credit questions. In order to obtain full credit for the partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work on a partial credit problem. THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.

For multiple choice problems, write the letter of your choice in the space provided below.

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1. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = b, \quad y'(0) = 3$$

For which value of $b$ is $\lim_{t \to +\infty} y(t) = 0$?

(a) there is no value for $b$ with the limit 0.
(b) 0.
(c) -1.
   Solution: $y = c_1 e^{-3t} + c_2 e^t, c_1 = \frac{b-3}{4}, c_2 = \frac{3b+3}{4}$ need $c_2 = 0$
(d) 1.

2. (5 points) Consider the following initial value problem:

$$yy' + y^2 = \frac{1}{2} + t, \quad y(0) = 0,$$

for $t \geq 0$. Let $y_1(t) = \sqrt{t}, \quad y_2(t) = -\sqrt{t}$.

Determine whether $y_1$ and $y_2$ are solutions of the initial value problem

(a) Both $y_1$ and $y_2$ are solutions.
   Solution: $y_1y'_1 + y_1^2 = \sqrt{t} \left(\frac{1}{2\sqrt{t}}\right) + (\sqrt{t})^2 = \frac{1}{2} + t$
   $y_2y'_2 + y_2^2 = -\sqrt{t} \left(\frac{1}{2\sqrt{t}}\right) + (-\sqrt{t})^2 = \frac{1}{2} + t$
(b) Neither $y_1$ nor $y_2$ are solutions.
(c) Only $y_1$ is a solution.
(d) Only $y_2$ is a solution.
3. (5 points) Consider the following first order Initial value problem:

\[ \sin(t)y' + y = \frac{\sin(t)}{\ln(t - 1)}, \quad y\left(\frac{2\pi}{3}\right) = 17. \]

Find the largest interval \( I \) where the solution to this problem is certain to exist.

(a) \( 0 < t < \pi \).
(b) \( 1 < t < \pi \).
(c) \( 2 < t < \pi \).

Solution: \( p(t) = \frac{1}{\sin(t)}, t \neq k\pi \) for any integer \( k \), \( q(t) = \frac{1}{\ln(t-1)}, t > 1, t \neq 2 \), initial \( t = \frac{2\pi}{3} > 2 \).
(d) \( \pi < t < 2\pi \).

4. (5 points) Consider the following first order ODE:

\[ \frac{dy}{dt} = -1 + \sqrt{1+y} \]

For which of the initial conditions i) \( y(1) = -1 \), ii) \( y(1) = 1 \) do we have the guaranteed existence and uniqueness of a (local) solution?

(a) For both of them.
(b) For neither of them.
(c) For i) only.
(d) For ii) only.

Solution: \( f(y) = -1 + \sqrt{1+y}, \quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{1+y}}, \quad y \neq -1 \).
5. (5 points) For which value of $\lambda$ are the functions $f_1(t) = t^2 - 1$ and $f_2(t) = t^2 + (1 - \lambda)t - \lambda$ linearly dependent?

(a) There is no such $\lambda$

(b) $\lambda = 0$

(c) $\lambda = 1$
   Solution: when $\lambda = 1$, $f_2(t) = t^2 - 1 = f_1(t)$, linearly dependent

(d) For all $\lambda$

6. (5 points) Let $y_1$ and $y_2$ be two independent solutions of the differential equation

$$ty'' + y' + e^ty = 0$$

Then the Wronskian $W(y_1, y_2)(t)$ of the equation equals (with $c$ a constant)

(a) $ce^{-t}$

(b) $ce^t$

(c) $ct^{-1}$
   Solution: $W(y_1, y_2)(t) = ce^{-\int p(t)dt} = ce^{-\int \frac{1}{t}dt} = ce^{-\ln t} = ct^{-1}$

(d) $ct$
7. (10 points) Consider the initial value problem
\[ y' - y = 3e^{-2t} + 1, \quad y(0) = y_0. \]

(a) Determine the integrating factor \( \mu(t) \).
Solution: \( \mu(t) = e^{\int p(t) dt} = e^{-t} \)

(b) Solve the above initial value problem.
Solution: multiply \( \mu(t) \) to the equation to get
\[ (e^{-t}y)' = 3e^{-3t} + e^{-t} \]
integrate on \( t \)
\[ e^{-t}y = -e^{-3t} - e^{-t} + C \]
\[ y = -e^{-2t} - 1 + Ce^t \]
substitute the initial conditions
\[ y_0 = -e^{-2\cdot0} - 1 + Ce^0 = -2 + C \]
\[ C = y_0 + 2 \]
The solution for the initial value problem
\[ y = -e^{-2t} - 1 + (y_0 + 2)e^t \]

(c) Figure out the value of \( y_0 \) such that the solution \( y(t) \) converges to a finite number as \( t \to \infty \).
Solution:
\[ \lim_{t \to \infty} e^{-2t} = 0 \]
\[ \lim_{t \to \infty} e^t = \infty \]
Only when \( y_0 = -2 \)
\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} -e^{-2t} - 1 = -1 \]
8. (15 points) A tank originally contains 100 gallon of water with 10 pound of salt in solution. Water containing 1 pound of salt per gallon is entering at a rate of 3 gal/min. The well stirred mixture flows out of the tank at the same rate. Let $Q(t)$ be the amount of salt in the tank at any time $t$.

(a) Set up an initial value problem for $Q(t)$.

Solution: \[
\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 3 \cdot 1 - 3 \frac{Q}{100}
\]

\[
\frac{dQ}{dt} + 3 \frac{Q}{100} = 3
\]

Initial condition: $Q(0) = 10$

(b) Solve the problem to find an expression for $Q(t)$.

Solution: Solve the equation using the method of integrating factor

\[
\mu(t) = e^{\frac{3}{100}t}
\]

\[
(e^{\frac{3}{100}t}Q)' = 3e^{\frac{3}{100}t}
\]

\[
e^{\frac{3}{100}t}Q = \int 3e^{\frac{3}{100}t}dt + C
\]

\[
e^{\frac{3}{100}t}Q = 100e^{\frac{3}{100}t} + C
\]

\[
Q = 100 + Ce^{\frac{3}{100}t}
\]

Substitute in the initial condition $Q(0) = 10$

\[
10 = 100 + Ce^{\frac{3}{100}} - 100 + C, \quad C = -90
\]

The solution is

\[
Q = 100 - 90e^{\frac{3}{100}t}
\]

(c) Find the limiting concentration of the salt in the tank.

Solution:

\[
\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} 100 - 90e^{\frac{3}{100}t} = 100
\]

The limiting concentration is

\[
\frac{\text{limiting salt amount}}{\text{tank volume}} = \frac{100}{100} = 1
\]
9. (15 points)  
(a) Solve the initial value problem

\[ y' = \frac{(3 + 2t)e^t}{\cos y}, \quad y(0) = 0. \]

Solution: The equation is separable

\[ (\cos y)dy = (3 + 2t)e^t \]

Integrate on both sides

\[ \sin y = 3e^t + 2(te^t - e^t) + C = e^t + 2te^t + C \]

Substitute the initial condition \( y(0) = 0 \)

\[ \sin 0 = e^0 + 2 \cdot 0 \cdot e^0 + C, \quad C = -1 \]

\[ \sin y = e^t + 2te^t - 1 \]

\[ y = \sin^{-1}(e^t + 2te^t - 1) \]

(b) Determine the interval in which the solution exists.

Solution: the definition interval for \( \sin^{-1}(e^t + 2te^t - 1) \) is

\[ -1 \leq e^t + 2te^t - 1 \leq 1 \]

But \( \cos y \neq 0 \), the definition interval is

\[ -1 < e^t + 2te^t - 1 < 1 \]
10. (10 pts) Consider the differential equation

\[ \frac{dy}{dt} = f(y) = y^2(4 - y^2) \]

(a) Find the equilibrium solutions of this differential equation. Determine the sign of \( f \) as a function of \( y \).

Solution: The equilibrium solutions are solved from \( f(y) = 0 \)

\[ y_1 = 2, \quad y_2 = 0, \quad y_3 = -2 \]

\[ y < -2, \quad f(y) < 0 \]
\[ -2 < y < 0, \quad f(y) > 0 \]
\[ 0 < y < 2, \quad f(y) > 0 \]
\[ y > 2, \quad f(y) < 0 \]

(b) Sketch the direction field of this equation and the graphs of a few representative solutions for \( t \geq 0 \).

(c) Classify the equilibrium solutions (asymptotically stable, unstable, or semi-stable).

Solution:
\[ y = 2, \text{ asymptotically stable} \]
\[ y = 0, \text{ semistable} \]
\[ y = -2, \text{ unstable} \]
11. (10 points)

(a) Solve the initial value problem

\[
\frac{d^2y}{dt^2} + \frac{dy}{dt} - 12y = 0
\]

with initial data \(y(0)=1\), \(y'(0)=2\)

Solution: characteristic equation

\[r^2 + r - 12 = 0, \quad r_1 = -4, \quad r_2 = 3\]

general solution

\[y(t) = c_1e^{-4t} + c_2e^{3t}\]

its derivative

\[y'(t) = -4c_1e^{-4t} + 3c_2e^{3t}\]

substitute initial conditions

\[1 = c_1 + c_2, \quad 2 = -4c_1 + 3c_2\]

the constants are \(c_1 = \frac{1}{7}\) and \(c_2 = \frac{6}{7}\)

the solution for the initial value problem

\[y(t) = \frac{1}{7}e^{-4t} + \frac{6}{7}e^{3t}\]

(b) Describe the behavior of this solution as \(t \to \infty\) and as \(t \to -\infty\).

Solution:

\[t \to \infty, \quad \frac{6}{7}e^{3t} \to \infty, \quad \frac{1}{7}e^{-4t} \to 0, \quad y \to \infty\]

\[t \to -\infty, \quad \frac{6}{7}e^{3t} \to 0, \quad \frac{1}{7}e^{-4t} \to \infty, \quad y \to \infty\]
12. (10 points)

(a) Find the general solution to

\[
\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 5 = 0
\]

Solution: characteristic equation

\[
r^2 + 2r + 5 = 0, \quad r_1 = -1 + 2i, \quad r_2 = -1 - 2i
\]

general solution

\[
y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t
\]

its derivative

\[
y'(t) = (-c_1 + 2c_2)e^{-t} \cos 2t + (-2c_1 - c_2)e^{-t} \sin 2t
\]

(b) Solve the initial value problem with initial data \(y(0) = 0,\ y'(0) = -3\).

Solution: substitute initial conditions

\[
0 = c_1, \quad -3 = 2c_2
\]

the constants are \(c_1 = 0, \ c_2 = -\frac{3}{2}\)

the solution is

\[
y(t) = -\frac{3}{2} e^{-t} \sin 2t
\]

(c) Sketch the graph of this solution for \(t > 0\)