Chapter 5. Exponential and Logarithmic Functions

5.1 Exponential Functions

The *exponential function* with base $a$ is defined by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$. Its domain is the set of all real numbers, and its range is the set of all positive numbers.

Graph of $f(x) = e^x$

The graphs of all other exponential functions $f(x) = a^x$ look similar, if $a > 1$. Each has $y$-intercept at $(0, a)$, no $x$-intercept, and $y = 0$ is the horizontal asymptote.
If the base $0 < a < 1$, the graph looks the reflection of the above in the $y$-axis. For example, take $a = 1/e < 1$, the graph of $f(x) = (1/e)^x = e^{-x}$ is below.

*Example:* The 3 familiar, and most often encountered, exponential functions are $y = 2^x$, $y = 10^x$, and $y = e^x$. But there are certainly $y = 7^x$ and even $y = \pi^x$. For base $a < 1$, the function is usually expressed in terms of its reciprocal: $y = (1/2)^x = (2^{-1})^x = 2^{-x}$.

*Example:* Use transformations to graph $y = 3^{(x + 1)} - 4$. 
Graph of $y = 3^{(x + 1)} - 4$

The Natural Exponential Function

The most commonly used base for exponential functions is the irrational number $e \approx 2.7182818\ldots$ One of the formulas that defines $e$ is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \ldots$$

The exponential function with base $e$, $f(x) = e^x$, is called the natural exponential function. But it is usually referred to simply as the exponential function.
Compound Interest

Compound interest is calculated by the formula

\[
A(t) = P \left(1 + \frac{r}{n}\right)^{nt}
\]

where \(A(t)\) is the account balance, \(P\) is the principal amount, \(r\) = interest rate per year (usually called annual percentage rate, or APR), \(t\) = number of years.

Continuous Compound Interest

If we let the frequency of compounding to become arbitrarily large (therefore, the time between successive compounding becomes arbitrarily small), the result is known as continuous compound interest.

Continuously compounded interest is calculated by the formula

\[
A(t) = Pe^{rt}
\]

where \(A(t)\) is the account balance, \(P\) is the principal amount, \(r\) = interest rate per year, \(t\) = number of years.

Comment: The exponential formula above is also used to model the growth of bacteria population (exponential growth) in a Petri dish; and, when \(r < 0\), the process of radioactive decay.
5.2 Logarithmic Functions

Exponential functions are one-to-one functions. Therefore, they each has an inverse function. The inverse of an exponential function is called a logarithmic function.

The logarithmic function with base $a$ is defined by

$$\log_a x = y \quad \text{if and only if} \quad a^y = x$$

That is $\log_a x$ is the exponent to which the base $a$ must be raised to get $x$. Its domain is the set of all positive numbers, and its range is the set of all real numbers.

Graph of $f(x) = \log_e x = \ln x$

As is the case for exponential functions, the graphs of all other logarithmic functions $f(x) = \log_a x$ look similar, for $a > 1$. Each has $x$-intercept at $(a, 0)$, no $y$-intercept, and $x = 0$ is a vertical asymptote.
Since $\log_a x$ and $a^x$ are inverse functions for each other, for each $a$, their graphs are therefore reflections of each other across the line $y = x$.

Graphs of $y = \ln x$ and $y = e^x$

Properties of Logarithms

1. $\log_a 1 = 0 \quad \text{b/c} \quad a^0 = 1$
2. $\log_a a = 1 \quad \text{b/c} \quad a^1 = a$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$, $x > 0$
Common Logarithm

The logarithm with base 10 is called the common logarithm and is denoted simply by “log”: \( \log x = \log_{10} x \).

It is defined by \( \log x = y \) if and only if \( 10^y = x \).

*Example:* \( \log 100 = 2; \quad \log .1 = -1; \quad \log \sqrt{10} = 1/2. \)

The common logarithm is sometimes used as scale of measurement of intensity where the range of values would otherwise be large. Examples are the decibel (dB) scale for loudness and the Richter magnitude scale for earthquake’s amplitude.

Natural Logarithm

The logarithm with base \( e \) is called the natural logarithm and is denoted by \( \ln \): \( \ln x = \log_e x \).

It is defined by \( \ln x = y \) if and only if \( e^y = x \).

*Example:* \( \ln e = 1; \quad \ln (1/e) = -1. \)
5.3 Laws of Logarithms

Laws of Logarithms

Let $a > 0$ be a constant, $a \neq 0$; $x$ and $y$ be positive numbers; $C$ be any real number.

1. $\log_a (xy) = \log_a x + \log_a y$

2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a (x^C) = C \log_a x$

Note: It is NOT true that

1'. $\log_a (x + y) = \log_a x + \log_a y$, nor

2'. $\frac{\log_a x}{\log_a y} = \log_a \left(\frac{x}{y}\right)$, nor

3'. $(\log_a x)^C = C \log_a x$

Example: (a) $\log 25 + \log 40 = \log (25 \cdot 40) = \log 1000 = 3$

(b) $4 \log 3 = \log 3^4 = \log 81$

(c) $e^{2 \ln(5)} = e^{\ln(5^2)} = e^{\ln(25)} = 25$

(d) $\log_2 8 - \log_5 125 = 3 - 3 = 0$
Example: Simplify
(a) $\ln (3^{\frac{1}{4}}e)$
(b) $2 \ln e^{(2x+1)} + \ln (4e^x)$
(c) $\log \sqrt[4]{\sqrt{yz}}$
(d) $\log (3x - 2) - 2 \log x + \frac{1}{3} \log (x^2 + 4)$

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This formula enables us to evaluate a logarithm of any base by converting it to an expression in terms of base $e$ or base 10.

Example: Evaluate $\log_7 100$.

$$\log_7 100 = \frac{\log_{10} 100}{\log_{10} 7} = \frac{2}{\log_{10} 7} \approx 2.367$$

Example: Simplify $(\log e)(\ln 10)$
5.4 Exponential and Logarithmic Equations

The two most important properties to remember when solving an equation containing exponential or logarithmic terms are their (inverse functions) cancellation properties: properties #3 and #4, section 5.2.

3. \( \log_a a^x = x \)

4. \( a^{\log_a x} = x, \quad x > 0 \)

Exponential Equations

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.

2. Take the logarithm of each side, use the Laws of Logarithms to cancel the exponential expression: “bring down the exponent”.

3. Solve for the variable.

Example: \( 5^{x+3} = 4 \)

\[
\log 5^{x+3} = \log 4
\]

\[
(x + 3) \log 5 = \log 4
\]

\[
x + 3 = \frac{\log 4}{\log 5}
\]

\[
x = \frac{\log 4}{\log 5} - 3
\]
Comment: It doesn’t really what base to use for the logarithm, since logarithms of different bases only differ from one other by a constant (see the Change of Base formula). In the above example, it would appear that logarithm with base 5 should be used. (Why?) We could have solved it using another base, as we did. Indeed, natural or common logarithm is almost always used, because of the ease of finding the actual numerical value of the solution using a calculator’s built-in logarithm functions.

Example: \[ 5^{2x} = e^{x-2} \]

Example: \[ x^2 e^{3x} = 4e^{3x} \]

Example: [Quadratic type] \[ e^{10x} + 3e^{5x} - 10 = 0 \]

Example: (#34) \[ e^x - 12e^{-x} - 1 = 0 \]
Logarithmic Equations

Guidelines for Solving Logarithmic Equations

1. Simplify, if necessary, and isolate the logarithmic expression on one side of the equation.

2. Write the equation in exponential form (or raise the same base to each side of the equation).

3. Solve for the variable.

Example: (#40) \( \log_3 (2 - x) = 3 \)

Example: \( \log x + \log(x - 21) = 2 \)
[Check your answer!]
Example: (49) \[ \log_9 (x - 5) + \log_9 (x + 3) = 1 \]

Example: \[ \ln(x + 3) + \ln(x - 3) = 0 \]

Example: [Half-Life] The half-life of a radioactive material is the length of time required for half of the initial quantity of material to have undergone radioactive decay. Find a formula expressing the half-life in terms of a radioactive material’s decaying constant \(-r\).