MATH 251 Work sheet / Things to know

Chapter 6

1. The Laplace transforms

The Laplace transform is an integral transformation that transforms a function of \( t \) (functions in \( t \)-space, or in engineering speak, in the time domain) to a function of a second independent variable \( s \) (in \( s \)-space, or the frequency domain). It is defined by the following definite integral (note that this is an improper integral):

\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} \, dt
\]

\( F(s) \) is called the transform or Laplace transform of \( f(t) \). This operation is one-to-one for any function \( f(t) \) that is continuous on \((0, \infty)\).

Ex. 6.1.1 Find the Laplace transform of \( f(t) = t \).

Q: What is the Laplace transform of \( f(t) = 0 \)?

2. Some properties of Laplace transforms

Linearity - Laplace transform is linear

\[
\mathcal{L}\{C_1 f(t) + C_2 g(t)\} = C_1 F_1(s) + C_2 F_2(s)
\]

The derivative of a Laplace transform

\[
F'(s) = \frac{d}{ds} F(s)
\]
The Laplace transform of a derivative

\[ \mathcal{L}\{f'(t)\} = \]

Therefore,

\[ \mathcal{L}\{f''(t)\} = \]

\[ \mathcal{L}\{f'''(t)\} = \]

etc.

3. Solving initial value problems

Before you proceed: review rules of partial fractions.

The method of the Laplace Transform is a whole “new system” of solving linear differential equations algebraically. The system works essentially the same way regardless the specifics of each linear equation in question (it does not require a separate step such as the method of undetermined coefficients for a nonhomogeneous equation).

What are the 3 stages of using the Laplace transform to solve a differential equation?
Ex. 6.3.1 (Ex. 3.3.1.a) \[ y'' + y' - 12y = 0, \quad y(0) = 0, \quad y'(0) = 2 \]

Without any modification, a nonhomogeneous linear equation can also be solved using the 3-step process:

Ex. 6.3.2 \[ y'' + 5y' + 6y = 3e^{4t}, \quad y(0) = 1, \quad y'(0) = -1 \]
Ex. 6.3.3 \[ y'' + 2y' + 10y = 5, \quad y(0) = 4, \quad y'(0) = 0 \]

Ex. 6.3.4 [\( F(s) \) with a quadratic or repeated factor in the denominator] Find the inverse Laplace transform of

(a) \[ F(s) = \frac{4s - 1}{s^2 + 9} \]

(b) \[ F(s) = \frac{4s - 1}{s^2 + 6s + 13} \]

(c) \[ F(s) = \frac{1}{s^2(s^2 + 1)} \]
4. Unit step functions

What is a unit step function, $u_c(t)$? What does its graph look like?

For our purpose of doing Laplace transform, $c$ must be positive. (Why?)

How about the difference $1 - u_c(t)$? What does its graph look like?

What is the Laplace transform of $u_c(t)$?

$$\mathcal{L}\{u_c(t)\} =$$

Ex. 6.4.1 Suppose $g(t) = 2t + 3u_1(t) - u_3(t)(t - 1) + u_5(t)(t^3 + 2t)$. What are $g(1)$, $g(2)$, $g(\pi)$ and $g(10)$?
5. **Piecewise continuous functions**

Understand how the unit step function, when used in a product form, can selectively make another function appear and/or disappear at a prescribed time.

<table>
<thead>
<tr>
<th>Product</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_c(t)f(t)$</td>
<td></td>
</tr>
<tr>
<td>$(1 - u_c(t))f(t)$</td>
<td></td>
</tr>
<tr>
<td>$(u_a(t) - u_b(t))f(t)$</td>
<td></td>
</tr>
</tbody>
</table>

A piecewise continuous function can be easily constructed (in a format suitable for solving differential equations) in terms of unit step functions by utilizing the above product expressions literally off-the-shelf.

How is this done?

---

**Ex. 6.5.1** Rewrite the given function in terms of step functions.

$$F(t) = \begin{cases} 
  te^{4t}, & 0 \leq t < 2 \\
  t^2 - 2t, & 2 \leq t < 3 \\
  5 - e^{-2t}, & t \geq 3 
\end{cases}$$
6. The translation theorems

The key to success for this chapter is to understand and master the 2 translation properties of the Laplace transforms, especially the Second Translation Theorem. It is critically important to know how to work the second translation theorem forward and (literally!) backward.

The First Translation Theorem

What does it say?

Ex. 6.6.1

(a) Given that \( \mathcal{L}\{t^4\} = \frac{24}{s^5} \), what is \( \mathcal{L}\{t^4 e^{-2t}\} \)?

(b) (Exam 2, summer 2007) Given that \( \mathcal{L}\{f(t)\} = F(s) \), what is \( \mathcal{L}\{e^{-5t}tf(t)\} \)?
The Second Translation Theorem

What does it say?

\[ \mathcal{L}\{u_c(t)f(t-c)\} = \]

Alternatively, and equivalently,

\[ \mathcal{L}\{u_c(t)g(t)\} = \]

The first formula is easier to use for the inverse transform, \( s \rightarrow t \); while the second is easier to use when transforming \( t \rightarrow s \).

Ex. 6.6.2 Find the inverse transform of

(a) \( F(s) = e^{-2s} \frac{2s + 1}{s^2 + 4} \)

(b) \( F(s) = e^{-3s} \frac{s}{s^2 - 4} \)

Ex. 6.6.3 Find the Laplace transforms of \( u_3(t)(t^2 - t + 4) \)

Ex. 6.6.4 Find the Laplace transforms of \( u_{\pi/4}(t)\cos(2t) \)

Ex. 6.6.5 Find the Laplace transforms of \( u_5(t)te^{-6t} \)
7. Solving equations with piecewise continuous forcing functions

For an equation with a piecewise continuous forcing function, we just need to incorporate the second translation theorem into the basic 3-step process of the Laplace transform method. Make sure you understand how to transform a piecewise continuous forcing function (at the start, \( t \to s \)), and later how to account for the effect of translation during the inverse transform (while recovering \( y(t) \) from \( Y(s) \), \( s \to t \)).

\[
Ex. 6.7.1 \quad y'' + 2y' + 5y = u_p(t), \quad y(0) = 3, \quad y'(0) = 0. 
\]
Ex. 6.7.2 \[ y'' + 9y = u_{2\pi}(t) \sin(2t), \quad y(0) = 0, \quad y'(0) = 10. \]

Ex. 6.7.3 \[ y'' + 4y' + 3y = t - 2u_{15}(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Like so many topics in this class: \textbf{Do many exercises!}
8. Impulse functions

How is the unit impulse function, $\delta(t)$, defined?

The impulse of $\delta(t)$ is located at $t = 0$. It can be translated, as usual. $\delta(t-c)$ has the impulse at $t = c$. (As before, for our purpose of doing Laplace transform, $c$ must be positive.)

What are the Laplace transforms of impulse functions?

$$\mathcal{L}\{\delta(t)\} =$$

$$\mathcal{L}\{\delta(t-c)\} =$$

And the product property (much easier to use than those of step functions)…

$$\mathcal{L}\{\delta(t-c)f(t)\} =$$

Notice that the impulse function also introduces the term $e^{-cs}$ into our transformed equation, just like a step function would, when using Laplace transform method. Therefore, an equation containing an impulsive forcing function is solved exactly like those with piecewise continuous forcing functions – all require the second translation theorem. The solution of such an equation will also be piecewise continuous, as a result, despite the discontinuous nature of the impulse function.
Ex. 6.8.1 \[ y'' + 16y = \delta(t - 2) - \delta(t - 5), \quad y(0) = 1, \quad y'(0) = -2. \]

Ex. 6.8.2 \[ y'' + 5y' + 6y = 2 - \delta(t - 4), \quad y(0) = 0, \quad y'(0) = 5. \]
Ex 6.8.3 (Exam 2, fall 2007) Solve the initial value problem
\[ y' + 2y = u_3(t) e^{t-3} + \delta(t - 3), \quad y(0) = 3. \]
Table of basic Laplace Transforms

\[ \mathcal{L} \{ f(t) \} = F(s) = \int_0^\infty e^{-st} f(t) \, dt \]

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
<td>( u_c(t) )</td>
<td>( \frac{e^{-cs}}{s} )</td>
</tr>
<tr>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
<td>( \delta(t) )</td>
<td>1</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
<td>( \delta(t-c) )</td>
<td>( e^{-cs} )</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( \frac{1}{s-a} )</td>
<td>( f'(t) )</td>
<td>( sF(s) - f(0) )</td>
</tr>
<tr>
<td>( t^n e^{at} )</td>
<td>( \frac{n!}{(s-a)^{n+1}} )</td>
<td>( f''(t) )</td>
<td>( s^2 F(s) - sf(0) - f'(0) )</td>
</tr>
<tr>
<td>( \cos bt )</td>
<td>( \frac{s}{s^2 + b^2} )</td>
<td>( (-t)^n f(t) )</td>
<td>( F^{(n)}(s) )</td>
</tr>
<tr>
<td>( \sin bt )</td>
<td>( \frac{b}{s^2 + b^2} )</td>
<td>( u_c(t)f(t-c) )</td>
<td>( e^{-cs} F(s) )</td>
</tr>
<tr>
<td>( e^{at} \cos bt )</td>
<td>( \frac{s-a}{(s-a)^2 + b^2} )</td>
<td>( e^{ct} f(t) )</td>
<td>( F(s-c) )</td>
</tr>
<tr>
<td>( e^{at} \sin bt )</td>
<td>( \frac{b}{(s-a)^2 + b^2} )</td>
<td>( \delta(t-c)f(t) )</td>
<td>( e^{-cs} f(c) )</td>
</tr>
</tbody>
</table>