Derivatives of Trigonometric Functions

The basic trigonometric limit:

**Theorem:** \[ \lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x} \] (\(x\) in radians)

**Note:** In calculus, unless otherwise noted, all angles are measured in radians, and not in degrees.

This theorem is sometimes referred to as the small-angle approximation because it really says that, for very small angles \(x\), \(\sin x \approx x\).

**Note:** Cosine behaves even better near 0, where \(\lim_{x \to 0} \cos x = 1\).

ex. Show that \(\lim_{x \to 0} \frac{\cos x - 1}{x} = 0\)

\[
\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{\cos x + 1} = \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}
\]

\[
= \lim_{x \to 0} \frac{-\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{\cos x + 1} = - \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{\cos x + 1} = -(1) \left( \frac{0}{1+1} \right) = 0
\]
ex. Evaluate \( \lim_{x \to 0} \frac{\sin 2x}{5x} \)

\[
\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{1}{5} \lim_{x \to 0} \frac{\sin 2x}{x} \cdot \frac{2}{2} = \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x}
\]

The idea above is to match the angle in the sine function with the denominator. We’ll then apply the basic trigonometric limit. To do so, first we substitute \( \theta = 2x \). Note that as \( x \) approaches 0, so does \( \theta \). Hence,

\[
\frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{5} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{2}{5} \cdot 1 = \frac{2}{5}
\]

ex. Evaluate \( \lim_{x \to 0} \frac{\sin 4x}{\sin 3x} \)

\[
\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} = \lim_{x \to 0} \frac{\sin 4x}{\sin 3x} \cdot \frac{x}{x} = \lim_{x \to 0} \frac{\sin 4x}{x} \cdot \frac{x}{\sin 3x} = \lim_{x \to 0} \frac{\sin 4x}{x} \cdot \lim_{x \to 0} \frac{x}{\sin 3x}
\]

Repeat the same trick as in the previous example, let \( \theta = 4x \) and \( \lambda = 3x \). Both \( \theta \) and \( \lambda \) approach 0 when \( x \) does. Then apply the theorem twice.

\[
= \lim_{x \to 0} \frac{\sin 4x}{x} \cdot \frac{4}{4} \cdot \lim_{x \to 0} \frac{x}{\sin 3x} \cdot \frac{3}{3} = \frac{4}{3} \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \lim_{x \to 0} \frac{3x}{\sin 3x}
\]

\[
= \frac{4}{3} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\lambda \to 0} \frac{\lambda}{\sin \lambda} = \frac{4}{3} \cdot 1 \cdot 1 = \frac{4}{3}
\]
In fact, after doing a few examples like those, we can see a (very nice) pattern. To sum it up:

Suppose $m$ and $n$ are nonzero real numbers, then

$$
\lim_{x \to 0} \frac{\sin mx}{nx} = \frac{m}{n}
$$

$$
\lim_{x \to 0} \frac{mx}{\sin nx} = \frac{m}{n}
$$

$$
\lim_{x \to 0} \frac{\sin mx}{\sin nx} = \frac{m}{n}
$$

(Trivially, we also have:

$$
\lim_{x \to 0} \frac{mx}{nx} = \frac{m}{n}.
$$)
ex. Evaluate \( \lim_{x \to 0} \frac{\tan 7x}{2x} \)

\[
\lim_{x \to 0} \frac{\tan 7x}{2x} = \lim_{x \to 0} \frac{1}{2x} \cdot \sin 7x = \frac{1}{2} \lim_{x \to 0} \frac{\sin 7x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin 7x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}
\]

\[
= \frac{1}{2} \cdot 7 \cdot \frac{1}{1} = \frac{7}{2}
\]

Recall that since \( \cos x \) is continuous everywhere, the direct substitution property applies, therefore,

\[
\lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{\lim_{x \to 0} \cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1
\]

Now, the main topic --

**Derivatives of Trigonometric Functions**

ex. What is the derivative of \( \sin x \)?

Start with the limit definition of derivative:

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{[\sin x \cos h + \sin h \cos x] - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\sin h \cos x}{h} = \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cdot \cos x
\]

\[
= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \cos x = \sin x \cdot (0) + (1) \cos x = \cos x
\]

Therefore, \( \frac{d}{dx} \sin x = \cos x \)
ex. Find the derivative of $\csc x$.

$$
\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = \frac{\sin x \left( \frac{d}{dx} 1 \right) - 1 \left( \frac{d}{dx} \sin x \right)}{(\sin x)^2} = \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x
$$

Therefore, $\frac{d}{dx} \csc x = -\csc x \cot x$

The complete list of derivatives of trigonometric functions:

1. $\frac{d}{dx} \sin x = \cos x$
2. $\frac{d}{dx} \cos x = -\sin x$
3. $\frac{d}{dx} \tan x = \sec^2 x$
4. $\frac{d}{dx} \sec x = \sec x \tan x$
5. $\frac{d}{dx} \cot x = -\csc^2 x$
6. $\frac{d}{dx} \csc x = -\csc x \cot x$
ex. Differentiate $f(x) = \sec x + 5 \csc x$

$f'(x) = \sec x \tan x + 5(-\csc x \cot x) = \sec x \tan x - 5 \csc x \cot x$

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ex. Differentiate $f(x) = x^2 \cos x - 2x \sin x - 3 \cos x$

$f'(x) = [x^2(-\sin x) + (2x) \cos x] - 2[x(\cos x) + (1)\sin x] - 3(-\sin x)$

$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2\sin x + 3\sin x$

$= -x^2 \sin x + \sin x$

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ex. Differentiate $s(t) = \frac{\sin t}{1 - \cos t}$

$s'(t) = \frac{(1 - \cos t)(\cos t) - (\sin t)(0 - (\sin t))}{(1 - \cos t)^2}$

$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2} = \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1 - \cos t)^2}$

$= \frac{\cos t - 1}{(1 - \cos t)^2} = \frac{-1}{(1 - \cos t)^2} = \frac{-1}{1 - \cos t} = \frac{1}{\cos t - 1}$
**Simple Harmonic Motion**  Suppose the oscillating motion (in meters) of a weight attached to a spring is described by the displacement function

\[ s(t) = 2 \cos t + \sin t \]

Find its velocity and acceleration functions, and its speed and acceleration at \( t = \pi/2 \) sec.

Velocity: \( v(t) = s'(t) = -2 \sin t + \cos t \)

Acceleration: \( a(t) = v'(t) = -2 \cos t - \sin t \)

Its speed when \( t = \pi/2 \) is

\[
| v(\pi/2) | = |-2 \sin (\pi/2) + \cos (\pi/2) | = |-2 + 0 | = 2 \text{ (m/sec)}
\]

Its acceleration at the same time is

\[
a(\pi/2) = -2 \cos (\pi/2) - \sin (\pi/2) = 0 - 1 = -1 \text{ (m/sec}^2)\]