First, we observe that

\[
\frac{n}{n^2 + 3n + 1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon
\]

\(\text{provided } n > \frac{1}{\varepsilon}\)

Thus, given any positive real number \(\varepsilon\), the number \(X = \frac{1}{\varepsilon}\) is such that

\[
\frac{n}{n^2 + 3n + 1} < \varepsilon \text{ for all } n \text{ larger than } X.
\]

In other words, the sequence \(\left\{\frac{n}{n^2 + 3n + 1}\right\}_{n=1}^\infty\) is null.

\[2\]
\[E \subset \mathbb{R} \quad E \text{ is bounded } \Rightarrow \text{E is bounded above by the sup}(E) \text{ exists.}\]

Set \(F = \left\{\sup(E) - x : x \in E\right\}\).

Clearly, \(F \subset \mathbb{R}\).

Moreover, \(E \neq \emptyset \Rightarrow \exists e \in E \Rightarrow \sup(e) - e \in F \Rightarrow F \neq \emptyset\).
Now $\sup(E)$ is an upper bound of $E$ so $\sup(E) \geq x$, $\forall x \in E$.

or, equivalently, $\frac{\sup(E) - x}{F} \geq 0$, $\forall x \in E$.

This shows that all elements of $F$ are $\geq 0$ or, in other words, that 0 is a lower bound of $F$.

$\emptyset \neq F \subseteq R$ \\
F bounded below $\rightarrow$ by the GLB property

$\inf(F)$ exists.

0 is a lower bound of $F$ \\
$\inf(F)$ is the greatest lower bound of $F$

So either $\inf(F) = 0$ or $\inf(F) > 0$.

We will show that $\inf(F) > 0$ is impossible.

Indeed, if $\inf(F)$ was strictly larger than 0, then we would have

$\frac{\sup(E) - x}{F} \geq \inf(F) > 0$, $\forall x \in E$

and then $\frac{\sup(E) - x}{F} \geq \inf(F)$, $\forall x \in E$ and $\inf(F) > 0$

(A) $\sup(E) - \inf(F) \geq x$, $\forall x \in E$  \\
(B) $\sup(E) > \sup(E) - \inf(F)$
(A) means that $\text{sup}(E) - \text{inf}(F)$ is an upper bound of $E$.

(B) means that $\text{sup}(E) - \text{inf}(F)$ is strictly smaller than the smallest upper bound of $E$.

So $\text{sup}(E) - \text{inf}(F)$ would be an upper bound of $E$ strictly smaller than the smallest upper bound of $E$.

This is impossible!

So the only possibility is that $\text{inf}(F) = 0$.

3 First, let's prove that $\sqrt{n^2+1} - n$ is positive then

We have

\[ 1 > 0 \]
\[ \implies n^2 + 1 > n^2 \quad (>0) \]
\[ \implies \sqrt{n^2+1} > n \quad (>0) \]
\[ \implies \sqrt{n^2+1} > n \implies \sqrt{n^2+1} - n > n - n \]
\[ \implies \sqrt{n^2+1} - n > 0 \]

Now, observe that

\[ |\sqrt{n^2+1} - n| = \sqrt{n^2+1} - n = \frac{\sqrt{n^2+1} - n}{\sqrt{n^2+1} + n} \]

\[ = \frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n} \]
I claim that \( \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n} \), \( \forall n \in \mathbb{N} \).

Indeed,

\[
1 > 0 \\
\Rightarrow \\
n^2 + 1 > n^2 > 0 \\
\Rightarrow \\
\sqrt{n^2 + 1} > n \\
\Rightarrow \\
\sqrt{n^2 + 1} + n > n + n \\
because n \in \mathbb{N} \\
\sqrt{n^2 + 1} + n > 2n > 0 \\
\Rightarrow \\
\frac{1}{\sqrt{n^2 + 1} + n} < \frac{1}{2n}.
\]

Therefore,

\[
|\sqrt{n^2 + 1} - n| = \frac{1}{\sqrt{n^2 + 1} + n} < \frac{1}{2n}, \forall n \in \mathbb{N}.
\]

We have shown that the sequence \( \left\{ \sqrt{n^2 + 1} - n \right\} \) is dominated by the sequence \( \left\{ \frac{1}{2n} \right\} \).

\[
|\sqrt{n^2 + 1} - n| < \frac{1}{2n}, \forall n \in \mathbb{N} \]

\( \left\{ \frac{1}{2n} \right\}_{n=1}^{\infty} \) is a null sequence \( \Rightarrow \) by the squeeze rule \( \left\{ \sqrt{n^2 + 1} - n \right\} \) is a null sequence.
Set \( E = \{ r \in \mathbb{Q} : r < b \} \)

We want to prove that \( \text{sup}(E) = b \).

**Claim #1:** \( E \) is not empty.

**Proof of Claim #1:** Consider the pair of real numbers \( b-1, b \).

By the density property of \( \mathbb{Q} \) in \( \mathbb{R} \), there exists a rational number \( q \) s.t. \( b-1 < q < b \).

Since \( q \in \mathbb{Q} \) and \( q < b \), \( q \) is an element of \( E \).

Therefore \( E \) is not empty.

**Claim #2:** \( \text{sup}(E) \) exists.

**Proof of Claim #2:** By the very definition of \( E \), the number \( b \) is an upper bound of \( E \).

Since \( E \) is a nonempty subset of the real line and is bounded above, the L.U.B. property guarantees that \( \text{sup}(E) \) exists.

\( b \) is an upper bound of \( E \) \( \Rightarrow \) \( \text{sup}(E) \leq b \);

\( \text{Sup}(E) \) is the smallest upper bound of \( E \) \( \Downarrow \) \( \text{Either } \text{sup}(E) < b \text{ or } \text{sup}(E) = b \)

**Claim #3:** \( \text{sup}(E) < b \) is impossible.
Proof of claim #3: If \( \text{sup}(E) \) was strictly smaller than \( b \), then, by density of \( \mathbb{Q} \) in \( \mathbb{R} \), there would exist a rational number \( \tilde{q} \) between \( \text{sup}(E) \) and \( b \):

\[
\text{sup}(E) < \tilde{q} < b
\]

However,

\[
\begin{align*}
q &\in \mathbb{Q} \\
\tilde{q} &< b
\implies \tilde{q} \in E \\
\implies \tilde{q} &\leq \text{sup}(E)
\end{align*}
\]

So that

\[
\text{sup}(E) < \tilde{q} \leq \text{sup}(E)
\]

Which is impossible.

We have shown that \( \text{sup}(E) < b \) leads to the impossible fact \( \text{sup}(E) < \text{sup}(E) \).

Since \( \text{sup}(E) < b \) is impossible and we must have \( \text{sup}(E) \leq b \), the only possibility is that \( \text{sup}(E) = b \).
If \((2 + \frac{1}{2})^\frac{1}{2}\) was rational, we could write it as a fraction \(\frac{p}{q}\) with \(p\) and \(q\) positive and having no common divisor \(> 1\).

But then:

\[
(2 + \frac{1}{2})^\frac{1}{2} = \frac{p}{q}
\]

\[
(\frac{5}{2})^\frac{1}{2} = \frac{p}{q}
\]

\[
\frac{5}{2} = (\frac{p}{q})^2
\]

\[
\frac{5}{2} = \frac{p^2}{q^2}
\]

\[
5q^2 = 2p^2
\]

2 is a divisor of \(5q^2\)

5 is a divisor of \(2p^2\)

5 is a divisor of \(p^2\)

5 is a divisor of \(p\)

5 is a divisor of \(2\)

5 is a divisor of \(2\)

Since 2 does not divide 5

2 is a divisor of \(q^2\)

5 is a divisor of \(p^2\)

2 is a divisor of \(q\)

5 is a divisor of \(p\)

2 is a divisor of \(q\)

5 is a divisor of \(5\)

2 is a divisor of \(2\)

\(q = 2b\) for some \(b \in \mathbb{N}\)

\(p = 5a\) for some \(a \in \mathbb{N}\)
Therefore \( 5q^2 = 2p^2 \) becomes

\[
5 \cdot (2b)^2 = 2 \cdot (5a)^2
\]

\[
2b^2 = 5a^2
\]

2 is a divisor of \( 5a^2 \) \[\iff\] 5 is a divisor of \( 2b^2 \)

\[
a = 2c \text{ for some } c \in \mathbb{N}
\]

\[
b = 5d \text{ for some } d \in \mathbb{N}
\]

It follows that

\[
p = 5a = 5(2c) = 10c
\]

and

\[
q = 2b = 2(5d) = 10d
\]

So \( p \) and \( q \) have the divisor 10 in common.

This is a contradiction.

We can therefore conclude that \( (2 + \frac{1}{2})^{\frac{1}{2}} \) cannot be written as a fraction \( \frac{p}{q} \) with ....

or, in other words, that \( (2 + \frac{1}{2})^{\frac{1}{2}} \) is not a rational number.
\[ S_n > 0, \quad \forall n \in \mathbb{N} \quad \text{(sequence of positive numbers)} \]
\[ S_{n+1} > S_n, \quad \forall n \in \mathbb{N} \quad \text{(increasing sequence)} \]

\[ \sigma_n = \frac{1}{n} (S_1 + S_2 + \cdots + S_n) \]

\[ \sigma_{n+1} - \sigma_n = \frac{S_1 + S_2 + \cdots + S_n + S_{n+1}}{n+1} - \frac{S_1 + S_2 + \cdots + S_n}{n} \]

\[ = \frac{nS_1 + nS_2 + \cdots + nS_n + nS_{n+1} - (n+1)S_1 - (n+1)S_2 - \cdots - (n+1)S_n}{(n+1)n} \]

\[ = \frac{n(S_{n+1} - S_1) - S_1 - S_2 - \cdots - S_n}{n(n+1)} \]

\[ = \left( \frac{S_{n+1} - S_1}{n(n+1)} \right) + \left( \frac{S_{n+1} - S_2}{n(n+1)} \right) + \cdots + \left( \frac{S_{n+1} - S_n}{n(n+1)} \right) \]

\[ \overset{N \to \infty}{\longrightarrow} > 0 \]

So we have proved that \( \sigma_{n+1} - \sigma_n > 0, \quad \forall n \in \mathbb{N} \).

\[ \sigma_{n+1} > \sigma_n, \quad \forall n \in \mathbb{N} \]

Thus, \( (\sigma_n) \) is also an increasing sequence.
\[
\left| \frac{\pi^n}{n!} \right| = \frac{\pi^n}{n!} = \frac{\pi}{1} \cdot \frac{\pi}{2} \cdot \frac{\pi}{3} \cdot \frac{\pi}{4} \cdot \frac{\pi}{5} \cdots \frac{\pi}{n-2} \cdot \frac{\pi}{n-1} \cdot \frac{\pi}{n} < \frac{\pi^4}{6} \cdot \frac{1}{n}
\]

A

\[-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}, \quad \forall x \in \mathbb{R}\]

\[
\arctan(x) < \frac{\pi}{2}, \quad \forall x \in \mathbb{R}
\]

\[
\arctan(\sqrt{n}) < \frac{\pi}{2}, \quad \forall n \in \mathbb{N}.
\]

\[
\left| \frac{\arctan(\sqrt{n})}{\sqrt{n}^{3/2}} \right| = \frac{\arctan(\sqrt{n})}{n^{3/2}} < \frac{\pi}{n^{3/2}} \leq \frac{\pi}{2} \cdot \frac{1}{n}
\]

B

\[
q_n = \left| \frac{\pi^n}{n!} + \frac{\arctan(\sqrt{n})}{\sqrt{n}^{3/2}} \right| < \left| \frac{\pi^n}{n!} \right| + \left| \frac{\arctan(\sqrt{n})}{\sqrt{n}^{3/2}} \right| < \frac{\pi^4}{6} \cdot \frac{1}{n} + \frac{\pi}{2} \cdot \frac{1}{n} \leq \left( \frac{\pi^4}{6} + \frac{\pi}{2} \right) \cdot \frac{1}{n}
\]

by A and B

\[
\text{triangle inequality}
\]
We have shown that the sequence \( \{a_n\} \) is dominated by the sequence \( \{(\pi^4 + \pi^2) \frac{1}{n}\} \). (c)

\( \frac{1}{n} \) is a null sequence \( \Rightarrow \) by the multiple rule \( \{(\pi^4 + \pi^2) \frac{1}{n}\} \) is a null sequence (D)

(c) & (D) \( \Rightarrow \) by the squeeze rule \( \{a_n\} \) is a null sequence.
\[
\left(\frac{(n+2)^2}{(n^2+2)} - 1\right) \left\lfloor \frac{(n+2)^2}{n^2+2} - 1 \right\rfloor = \left(\frac{(n+2)^2 - (n^2+2)}{n^2+2}\right) = \frac{n^2+4n+4 - n^2 - 2}{n^2+2}
\]

\[
\frac{(n+2)^2}{n^2+2} - 1 = \frac{4n+2}{n^2+2} = \frac{4}{n} + \frac{2}{n^2}
\]

\[
\frac{1}{n^2+2} < \frac{1}{n^2} \Rightarrow \frac{n^2}{n^2+2} < 1 \Rightarrow 0 < 2
\]

We have just proved that \(\left(\frac{(n+2)^2}{n^2+2} - 1\right)\) is dominated by \(\left(\frac{4}{n} + \frac{2}{n^2}\right)\)

\[
\left(\frac{1}{n}\right) \text{ null } \Rightarrow \left(\frac{4}{n}\right) \text{ null}
\]

\[
\left(\frac{1}{n^2}\right) \text{ null } \Rightarrow \left(\frac{2}{n^2}\right) \text{ null}
\]

\[
\left(\frac{4}{n}\right) \text{ null } \Rightarrow \left(\frac{4}{n} \cdot \frac{1}{n^2}\right) \text{ null}
\]

\[
\left(\frac{2}{n^2}\right) \text{ null } \Rightarrow \left(\frac{2}{n^2} \cdot \frac{1}{n^2}\right) \text{ null}
\]

\[
\left(\text{squeeze rule}\right) \frac{(n+2)^2}{n^2+2} - 1 \text{ is null.}
\]