Review Problems

Problem 1. Show that
\[ \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } 2 - \epsilon < \frac{2n + 1}{n + 2} < 2 + \epsilon, \text{ for } n \geq N. \]

Problem 2. Prove by induction that \( n^3 + 5n \) is divisible by 6 for all \( n \in \mathbb{N} \).

Problem 3. The Fibonacci numbers from a sequence \( \{F_n\} \) defined by \( F_1 = F_2 = 1 \) and \( F_{n+2} = F_{n+1} + F_n \). Show that
\[ F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}, \quad \forall n \in \mathbb{N}. \]

Problem 4. Let \( S \) and \( T \) be two nonempty bounded subsets of \( \mathbb{R} \) with \( S \subset T \). Prove that
\[ \inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T). \]

Problem 5. Let \( y \) be a nonnegative real number. Prove that there exists a real number \( x \) such that \( x^2 = y \). (Hint: find a subset \( S \) of \( \mathbb{R} \) such that \( (\sup S)^2 = y \).)

Problem 6. Prove that
\[ \frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}, \quad \forall x, y \in \mathbb{R}. \]
(Hint: consider the cases \( xy \geq 0 \) and \( xy < 0 \) separately.)

Problem 7. Prove that if a sequence \( \{x_n\} \) converges to a real number \( l \), then the sequence \( \{|x_n|\} \) converges to \( |l| \). Is the converse true?

Problem 8. Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences of real numbers such that \( x_n \leq y_n \) \( (\forall n \in \mathbb{N}) \), \( \{x_n\} \) is increasing, and \( \{y_n\} \) is decreasing. Prove that \( \{x_n\} \) and \( \{y_n\} \) are convergent and that \( \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n \).

Problem 9. Is the sequence \( \{x_n\} \) given by
\[ x_n = \frac{n^2}{\sqrt{n^6 + 1}} + \frac{n^2}{\sqrt{n^6 + 2}} + \cdots + \frac{n^2}{\sqrt{n^6 + n}} \]
convergent or divergent?

Problem 10. Let \( \{x_n\} \) be a sequence of real numbers with \( x_n \neq 0 \) \( (\forall n \in \mathbb{N}) \). Assume that
\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = l. \]
(a) Prove that, if \( |l| < 1 \), then \( \lim_{n \to \infty} x_n = 0 \).
(b) Prove that, if \( |l| > 1 \), then the sequence \( \{x_n\} \) diverges.

Problem 11. Consider the sequence \( x_n = n^{1/n} - 1 \).
(a) Using the binomial expansion of \( (1 + x_n)^n \), prove that \( 0 \leq x_n \leq \frac{\sqrt{5}}{n-1} \) for \( n \geq 2 \).
(b) Prove that \( \lim_{n \to \infty} n^{\frac{1}{n}} = 1 \). (Hint: find \( \lim_{n \to \infty} x_n \).

**Problem 12.**

(a) Find a (nonempty) subset \( S \) of \( \mathbb{Q} \) which is bounded above but does not have a least upper bound in \( \mathbb{Q} \). Justify.

(b) Since \( \emptyset \neq S \subset \mathbb{R} \), your set \( S \) must have a least upper bound in \( \mathbb{R} \). Find it.

(c) Prove that \( T = \{ t \in \mathbb{R} : t \geq s, \forall s \in S \} \) is an interval of \( \mathbb{R} \) unbounded above.

(d) Does \( T \) have a minimum? Why?

**Problem 13.** Assuming \( x \geq 1 \), prove that

\[
\left( \frac{2 \sqrt{x} - 1}{x^2} \right)^n = \left( 1 - \left( 1 - \frac{1}{\sqrt{x}} \right)^2 \right)^n 
\]

\[
\left( \frac{2 \sqrt{x} - 1}{x^2} \right)^n \leq 1
\]

and, using Bernoulli’s inequality

\[
1 + ny \leq (1 + y)^n, \quad \text{for } y \geq -1 \text{ and } n \in \mathbb{N},
\]

show that

\[
1 - n \left( 1 - \frac{1}{\sqrt{x}} \right)^2 \leq \left( 1 - \left( 1 - \frac{1}{\sqrt{x}} \right)^2 \right)^n \]

\[
n(\sqrt{x} - 1) \leq x - 1.
\]

Then, using (1), (3) and (4), prove that

\[
\left( \frac{2 \sqrt{x} - 1}{x^2} \right)^n \geq 1 - \frac{(x - 1)^2}{n \sqrt{x}^2}.
\]

Finally, deduce from (2) and (5) that

\[
\lim_{n \to \infty} (2 \sqrt{x} - 1)^n = x^2.
\]

**Problem 14.**

(a) Prove that

\[
0 \leq e^x - 1 \leq 2x, \quad \forall x \in [0, 1].
\]

Hint: if \( x \in [0, 1] \), then \( 0 \leq \sum_{k=2}^{\infty} \frac{x^{k-1}}{k!} \leq \sum_{k=2}^{\infty} \frac{1}{k!} \leq e - 2 \leq 1.\)

(b) Deduce that

\[
0 \leq \sqrt[n]{n} - 1 \leq 2 \frac{\ln(n)}{n}, \quad \forall n \in \mathbb{N}.
\]

(c) Prove that

\[
\lim_{n \to \infty} n(\sqrt[n]{n} - 1)^2 = 0.
\]

Hint: \( \lim_{y \to \infty} \frac{(\ln y)^2}{y} = \lim_{x \to \infty} \frac{x^2}{e^x} \)

(d) Prove that

\[
\lim_{n \to \infty} \frac{(2 \sqrt[n]{n} - 1)^n}{n^2} = 1.
\]
Hint: we know from Problem 13 that
\[ 1 - n \left( 1 - \frac{1}{\sqrt{n}} \right)^2 \leq \frac{(2\sqrt{n} - 1)^n}{n^2} \leq 1, \quad \forall n \in \mathbb{N}. \]

**Problem 15.** Let \( \alpha \) be a positive real number larger than 1. Prove that the sequence \( \{x_n\} \) defined by \( x_1 = 1 \) and
\[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) \]
is convergent and find its limit.

**Problem 16.**
(a) Does the series \( \sum_{n=1}^{\infty} \left( -1 \right)^n \sqrt{n} \) converge or diverge.
(b) Does the series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n} \) converge or diverge.

**Problem 17.** Is the series
\[ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \]
convergent or divergent?

**Problem 18.**
(a) State the comparison test for series of real numbers.
(b) Prove that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.
(c) Prove that the series \( \sum_{n=1}^{\infty} \frac{2013}{3\sqrt{n}} \) diverges.

**Problem 19.** Write down explicitly what it means for a sequence \( \{a_n\} \) to be bounded.

**Problem 20.** Prove that a convergent sequence \( \{a_n\}_{n=1}^{\infty} \) of real numbers is necessarily bounded.

**Problem 21.** Write down explicitly what it means for a sequence \( \{a_n\} \) to converge to a real number \( l \).

**Problem 22.** Let \( \{a_n\} \) and \( \{b_n\} \) be two convergent sequences. Prove that, if \( a_n \leq b_n \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

**Problem 23.** Write down explicitly what it means for a sequence \( \{a_n\} \) of real numbers to tend to \( \infty \).

**Problem 24.** Prove that if a sequence \( \{a_n\}_{n=1}^{\infty} \) of real numbers tends to \( \infty \) then so does any subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \).

**Problem 25.** Prove that, if a \( \{a_{n_k}\}_{k=1}^{\infty} \) is a subsequence of an increasing sequence of real numbers \( \{a_n\}_{n=1}^{\infty} \), then \( \{a_{n_k}\}_{k=1}^{\infty} \) is itself an increasing sequence.

**Problem 26.** Prove that
\[ \sum_{k=0}^{n} q^k = \frac{q^{n+1} - 1}{q - 1} \quad \text{if } q \neq 1. \]
Problem 27. For which values of the real numbers \( a \) and \( r \) is the geometric series \( \sum_{k=0}^{\infty} ar^k \) convergent? Justify your answer.

Problem 28. (a) Give a complete statement of the Monotone Convergence Theorem for a sequence of real numbers.

(b) Give a complete statement of the Bolzano–Weierstrass Theorem.

Problem 29. (a) Let \( \{a_n\} \) be a sequence of real numbers. Prove that, if the series \( \sum_{n=1}^{\infty} a_n \) converges, the sequence \( \{a_n\} \) must converge to 0.

(b) Find a null sequence \( \{b_n\} \) such that \( \sum_{n=1}^{\infty} b_n \) diverges.

Problem 30. Let \( s_n \) denote the sum of the first \( n \) terms of the series of real numbers \( a_1 + a_2 + a_3 + a_4 + a_5 + \cdots \). If the sequence \( \{s_n\}_{n=1}^{\infty} \) is increasing and bounded below, what can one say about the convergence behavior of the series \( \sum_{n=1}^{\infty} a_n \)?