Problem 1. Verify that
\[ u(x, t) = \begin{cases} 
  u_l & x < st \\
  u_r & x > st 
\end{cases} \]
where \( s = (u_l + u + u_r)/2 \)
is a weak solution to the Burgers’ equation
\[ u_t + uu_x = 0, \quad u(x, 0) = \begin{cases} 
  u_l & x < 0 \\
  u_r & x > 0 
\end{cases}. \]

Problem 2. Show that the viscous Burgers’ equation
\[ u_t + uu_x = \varepsilon u_{xx} \]
has a travelling wave solution of the form \( u^\varepsilon(x, t) = w(x - st) \) by deriving an ODE for \( w \) and verifying that this ODE has solutions of the form
\[ w(y) = u_r + \frac{1}{2}(u_l - u_r)[1 - \tanh((u_l - u_r)y/(4\varepsilon))] \]
with \( s = (u_l + u_r)/2 \). Note that \( w(y) \to u_l \) as \( y \to -\infty \) and \( w(y) \to u_r \) as \( y \to +\infty \). Use for example Matlab to plot this solution and observe how it varies as \( \varepsilon \to 0 \).

Problem 3. There are infinitely many weak solutions to the Riemann problem of Burgers’ equation
\[ u_t + uu_x = 0, \quad u(x, 0) = \begin{cases} 
  u_l & x < 0 \\
  u_r & x > 0 
\end{cases} \]
when \( u_l < u_r \). Show, for example, that
\[ u(x, t) = \begin{cases} 
  u_l & x < s_m t \\
  s_m t \leq x \leq u_m t & x/ t \\
  u_r & x > u_r t 
\end{cases} \]
is a weak solution for any \( u_m \) with \( u_l \leq u_m \leq u_r \) and \( s_m = (u_l + u_m)/2 \). Sketch the characteristics for this solution.

Problem 4. Consider the Riemann problem
\[ u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} 
  u_l & x < 0 \\
  u_r & x > 0 
\end{cases}, \]
where \( f'' > 0 \) (convex) and \( u_l < u_r \). Show that the rarefaction wave solution is given by
\[ u(x, t) = \begin{cases} 
  u_l & x < f'(u_l)t \\
  v(x/t) & f'(u_l)t \leq x \leq f'(u_r)t \\
  u_r & x > f'(u_r)t 
\end{cases} \]
where \( v(\xi) \) is the solution to \( f'(v(\xi)) = \xi \).