Alexandrov meets Lott–Villani–Sturm

Anton Petrunin

Abstract

Here I show compatibility of two definition of generalized curvature bounds — the lower bound for sectional curvature in the sense of Alexandrov and lower bound for Ricci curvature in the sense of Lott–Villani–Sturm.

Introduction

Let me denote by $\text{CD}[m, \kappa]$ the class of metric-measure spaces which satisfy a weak curvature-dimension condition for dimension $m$ and curvature $\kappa$ (see preliminaries). By $\text{Alex}^m[\kappa]$, I will denote the class of all $m$-dimensional Alexandrov spaces with curvature $\geq \kappa$ equipped with the volume-measure (so $\text{Alex}^m[\kappa]$ is a class of metric-measure spaces).

Main theorem. $\text{Alex}^m[0] \subset \text{CD}[m, 0]$.

The question appears first in [Lott–Villani, 7.48]. In [Villani], it is formulated more generally: $\text{Alex}^m[\kappa] \subset \text{CD}[m, (m-1)\kappa]$. The later statement can be proved, along the same lines, but I do not write it down.

About the proof. The idea of the proof is the same as in the Riemannian case (see [CMS, 6.2] or [Lott–Villani, 7.3]). One only needs to extend certain calculus to Alexandrov spaces. To do this, I use the same technique as in [Petrunin 03]. I will illustrate the idea on a very simple problem.

Let $M$ be a 2-dimensional non-negatively curved Riemannian manifold and $\gamma_\tau: [0, 1] \to M$ be a continuous family of unit-speed geodesics such that

$$|\gamma_\tau(t_0) \gamma_\tau(t_1)| \geq |t_1 - t_0|.$$  

Set $\ell(t)$ to be the total length of curve $\sigma_t: \tau \mapsto \gamma_\tau(t)$. Then $\ell(t)$ is a concave function — that is easy to prove.

Now, assume you have $A \in \text{Alex}^2[0]$ instead of $M$ and a non-continuous family of unit-speed geodesics $\gamma_\tau(t)$ which satisfies ①. Define $\ell(t)$ as the 1-dimensional Hausdorff measure of image of $\sigma_t$. Then $\ell$ is also concave.

Here is an idea how one can proceed; it is not the simplest one but the one which admits a proper generalization. Consider two functions $\psi = \text{dist}_{\text{im} \sigma_0}$ and $\varphi = \text{dist}_{\text{im} \sigma_1}$. Note that geodesics $\gamma_\tau(t)$ are also gradient curves of $\psi$ and $\varphi$. This implies that $\Delta \varphi + \Delta \psi$ vanishes almost everywhere on the image of the map $(\tau, t) \to \gamma_\tau(t)$ (the Laplasians $\Delta \varphi$ and $\Delta \psi$ are Radon sign-measures). Then the result follows from the second variation formula from [Petrunin 98] and calculus on Alexandrov spaces developed in [Perelman].
Remark. Although CD\([m, \kappa]\) is a very natural class of metric-measure spaces, some basic tools in Ricci comparison can not work there in principle. For instance, there are CD\([m, 0]\)-spaces which do not satisfy the Abresch–Gromoll inequality, (see [AG]). Thus, one has to modify the definition of the class CD\([m, \kappa]\) to make it suitable for substantial applications in Riemannian geometry.

I’m grateful to A. Lytchak and C. Villani, for their help.

1 Preliminaries

Prerequisite. The reader is expected to be familiar with basic definitions and notions of optimal transport theory as in [Villani], measure theory on Alexandrov spaces from [BGP], DC-structure on Alexandrov’s spaces from [Perelman] and technique and notations of gradient flow as in [Petrunin 07].

What needs to be proved. Let me recall the definition of class CD\([m, 0]\) only — it is sufficient for understanding this paper. The definition of CD\([m, \kappa]\) can be found in [Villani, 29.8].

Similar definitions were given in [Lott–Villani] and [Sturm]. The idea behind these definitions — convexity of certain functionals in the Wasserstein space over a Riemannian manifold, appears in [Otto–Villani], [CMS], [Sturm–v. Renesse]. In the Euclidean context, this notion of convexity goes back to [McCann]. More on the history of the subject can be found in [Villani].

For a metric-measure space \(X\), I will denote by \(|xy|\) the distance between points \(x, y\in X\) and by \(\text{vol} E\) the distinguished measure of Borel subset \(E \subset X\) (I will call it volume). Let us denote by \(P^2 X\) the set of all probability measures with compact support in \(X\) equipped with Wasserstein distance of order 2, see [Villani, 6.1].

Further, we assume \(X\) is a proper geodesic space; in this case \(P^2 X\) is geodesic.

Let \(\mu\) be a probability measure on \(X\). Denote by \(\mu^r\) the absolutely continuous part of \(\mu\) with respect to volume. I.e. \(\mu^r\) coincides with \(\mu\) outside a Borel subset of volume zero and there is a Borel function \(\rho: X \to \mathbb{R}\) such that \(\mu^r = \rho \cdot \text{vol}\).

Define

\[
U_{m, \mu} \overset{\text{def}}{=} \int_X \rho^{1 - \frac{m}{2}} \cdot d\text{vol} = \int_X \frac{1}{\sqrt{\rho}} \cdot d\mu^r.
\]

Then \(X \in \text{CD}[m, 0]\) if the functional \(U_m\) is concave on \(P^2 X\); i.e for any two measures \(\mu_0, \mu_1 \in P^2 X\), there is a Borel function \(\rho: X \to \mathbb{R}\) such that \(\mu^r = \rho \cdot \text{vol}\).

1 Calculus in Alexandrov spaces. Let \(A \in \text{Alex}^m[k]\) and \(S \subset A\) be the subset of singular points; i.e. \(x \in S\) iff its tangent space \(T_x\) is not isometric to Euclidean \(m\)-space \(\mathbb{E}^m\). The set \(S\) has zero volume ([BGP, 10.6]). The set of regular points \(A \setminus S\) is convex ([Petrunin 98]); i.e. any geodesic connecting two regular points consists only of regular points.

According to [Perelman], if \(f: A \to \mathbb{R}\) is a semiconcave function and \(\Omega \subset A\) is an image of a DC_0-chart, then \(\partial_k f\) and components of metric tensor \(g^{ij}\) are functions of locally bounded variation which are continuous in \(\Omega \setminus S\).

Further, for almost all \(x \in A\) the Hessian of \(f\) is well defined. I.e. there is a subset of full measure \(\text{Reg} f \subset A \setminus S\) such that for any \(p \in \text{Reg} f\) there is a
bi-linear form\(^1\) Hess\(_p\) \(f\) on \(T_p\) such that
\[
f(q) = f(p) + d_p f(v) + \text{Hess}_p f(v, v) + o(|v|^2),
\]
where \(v = \log_p q\). Moreover, the Hessian can be found using standard calculus in the DC\(_0\)-chart. In particular,
\[
\text{Trace Hess } f \overset{\text{\(\sim\)}}{=} \frac{\partial_i (\det g \cdot g^{ij} \cdot \partial_j f)}{\det g}
\]

The following is an extract from the second variation formula [Petrunin 98, 1.1B] reformulated with formalism of ultrafilters. Let \(\omega\) be a nonprincipal ultrafilter on natural numbers, \(A \in \text{Alex}^m[0]\) and \([pq]\) be a minimizing geodesic in \(A\) which is extendable beyond \(p\) and \(q\). Assume further that one of (and therefore each) of the points \(p\) and \(q\) is regular. Then there is a model configuration \(\tilde{p}, \tilde{q} \in E^m\) and isometries \(t_p : T_p A \to T_{\tilde{p}} E^m, t_q : T_q A \to T_{\tilde{q}} E^m\) such that
\[
|\exp_{\tilde{p}}(\frac{1}{n} v) \exp_{\tilde{q}}(\frac{1}{n} w)| \leq |\exp_{\tilde{p}} \circ t_p(\frac{1}{n} v) \exp_{\tilde{q}} \circ t_q(\frac{1}{n} w)| + o(n^2)
\]
for \(\omega\)-almost all \(n\) (once the left-hand side is well defined).

If \(\tilde{\tau} : T_{\tilde{p}} \to T_{\tilde{q}}\) is the parallel translation in \(E^m\), then the isometry \(\tau : T_p \to T_q\) which satisfy identity \(t_q \circ \tau = \tilde{\tau} \circ t_p\) will be called the “parallel transportation” from \(p\) to \(q\).

**Laplacians of semiconcave functions.**

Here are some facts from [Petrunin 03].

Given a function \(f : A \to \mathbb{R}\), define its Laplacian \(\Delta f\) to be a Radon sign-measure which satisfies the following identity
\[
\int_A u \cdot d\Delta f = -\int_A \langle \nabla u, \nabla f \rangle \cdot d\text{vol}
\]
for any Lipschitz function \(u : A \to \mathbb{R}\).

### 1.1. Claim

Let \(A \in \text{Alex}^m[\kappa]\) and \(f : A \to \mathbb{R}\) be \(\lambda\)-concave Lipschitz function. Then Laplacian \(\Delta f\) is well defined and
\[
\Delta f \leq m \lambda \text{vol}.
\]
In particular, \(\Delta^s f\) — the singular part \(\Delta f\) is negative.

Moreover,
\[
\Delta f = \text{Trace Hess } f \cdot \text{vol} + \Delta^s f.
\]

**Proof.** Let us denote by \(F_t : A \to A\) the \(f\)-gradient flow for time \(t\).

Given a Lipschitz function \(u : A \to \mathbb{R}\), consider family \(u_t(x) = u \circ F_t(x)\). Clearly, \(u_0 \equiv u\) and \(u_t\) is Lipschitz for any \(t \geq 0\). Further, for any \(x \in A\) we have \(\|\frac{d}{dt} u_t(x)\|_{t=0} \leq \text{Const}\). Moreover
\[
\frac{d}{dt} u_t(x)\big|_{t=0} \overset{\text{\(\sim\)}}{=} d_x u(\nabla_x f) \overset{\text{\(\sim\)}}{=} \langle \nabla_x u, \nabla_x f \rangle.
\]

Further,
\[
\int_A u_t \cdot d\text{vol} = \int_A u \cdot d(F_t \# \text{vol}),
\]

\(^1\)Note that \(p \in A\setminus S\), thus \(T_p\) is isometric to Euclidean \(m\)-space.
where # stands for push-forward. Since $|F_t(x)F_t(y)| \leq e^{\lambda t} |xy|$ (see [Petrunin 07, 2.1.4(i)]), for any $x, y \in A$ we have

$$F_t\# \text{vol} \geq \exp(-m\lambda t) \cdot \text{vol}.$$  

Therefore, for any non-negative Lipschitz function $u: A \to \mathbb{R}$,

$$\int_A u_t \cdot d\text{vol} = \int_A u \cdot d(F_t\# \text{vol}) \geq \exp(-m\lambda t) \int_A u \cdot d\text{vol}.$$  

Therefore

$$\int_A (\nabla u, \nabla f) \cdot d\text{vol} = \frac{dt}{\lambda} \left| u_t \cdot d\text{vol} \right|_{t=0} \geq -m\lambda \cdot \int_A u \cdot d\text{vol}.$$  

I.e. there is a Radon measure $\chi$ on $A$, such that

$$\int_A u \cdot d\chi = \int_A [(\nabla u, \nabla f) + m\lambda u] \cdot d\text{vol}.$$  

Set $\Delta f = -\chi + m\lambda$, it is a Radon sign-measure and $\chi = -\Delta f + m\lambda \geq 0$.

To prove the second part of theorem, assume $u$ is a non-negative Lipschitz function with support in a DC$_0$-chart $U \to A$, where $U \subset \mathbb{R}^m$ is an open subset. Then

$$\int_U (\nabla u, \nabla f) = \det g \cdot g^{ij} \cdot \partial_i u \cdot \partial_j f \cdot dx^1 \cdot dx^2 \cdot \cdots \cdot dx^m =$$

$$=-\int_U u \cdot \partial_t (\det g \cdot g^{ij} \cdot \partial_j f) \cdot dx^1 \cdot dx^2 \cdot \cdots \cdot dx^m,$$

Thus

$$\Delta f = \partial_t (\det g \cdot g^{ij} \cdot \partial_j f) \cdot dx^1 \cdot dx^2 \cdot \cdots \cdot dx^m \overset{\text{Trace Hess}}{=} \text{Trace Hess } f.$$  

Gradient curves. Here I extend the notion of gradient curves to families of functions, see [Petrunin 07] for all necessary definitions.

Let $I$ be an open real interval and $\lambda: I \to \mathbb{R}$ be a continuous function. A one parameter family of functions $f_t: A \to \mathbb{R}$, $t \in I$ will be called $\lambda(t)$-concave if the function $(t, x) \mapsto f_t(x)$ is locally Lipschitz and $f_t$ is $\lambda(t)$-concave for each $t \in I$.

We will write $\alpha^\pm(t) = \nabla f_t$ if for any $t \in I$, the right/left tangent vector $\alpha^\pm(t)$ is well defined and $\alpha^\pm(t) = \nabla \alpha(t)f_t$. The solutions of $\alpha^+(t) = \nabla f_t$ will be also called $f_t$-gradient curves.

The following is a slight generalization of [Petrunin 07, 2.1.2&2.2(2)]; it can be proved along the same lines.

1.2. Proposition-definition. Let $A \in \text{Alex}^m(i]$, $I$ be an open real interval, $\lambda: I \to \mathbb{R}$ be a continuous function and $f_t: A \to \mathbb{R}$, $t \in I$ be $\lambda(t)$-concave family.

Then for any $x \in A$ and $t_0 \in I$ there exists an $f_t$-gradient curve $\alpha$ which is defined in a neighborhood of $t_0$ and such that $\alpha(t_0) = x$.

Moreover, if $\alpha, \beta: I \to A$ are $f_t$-gradient then for any $t_0, t_1 \in I$, $t_0 \leq t_1$,

$$|\alpha(t_1)\beta(t_1)| \leq L.|\alpha(t_0)\beta(t_0)|,$$
where \( L = \exp \left( \int_{t_0}^{t_1} \lambda(t) \cdot dt \right) \).

Note that the above proposition implies that the value \( \alpha(t_0) \) of an \( f_t \)-gradient curve \( \alpha(t) \) uniquely determines it for all \( t \geq t_0 \) in \( I \). Thus we can define \( f_t \)-gradient flow — a family of maps \( F_{t_0,t_1} : A \to A \) such that

\[
F_{t_0,t_1}(\alpha(t_0)) = \alpha(t_1) \quad \text{if} \quad \alpha^+(t) = \nabla f_t.
\]

**1.3. Claim.** Let \( f_t : A \to \mathbb{R} \) be a \( \lambda(t) \)-concave family and \( F_{t_0,t_1} \) be \( f_t \)-gradient flow. Let \( E \subset A \) be a bounded Borel set. Fix \( t_1 \) and consider the function

\[
v(t) = \text{vol} F_{t_0,t_1}^{-1}(E). \]

Then

\[
v\big|_{t_1} = \int_{t_0}^{t_1} \Delta f_t \left[ F_{t_0,t_1}^{-1}(E) \right] \cdot dt.
\]

**Proof.** Let \( u : A \to \mathbb{R} \) be a Lipschitz function with compact support. Set \( u_t = u \circ F_{t_0,t_1} \). Clearly all \( (x,t) \mapsto u_t(x) \) is locally Lipschitz. Thus, the function

\[
w_u : t \mapsto \int_A u_t \cdot d\text{vol}
\]

is locally Lipschitz. Further

\[
w_u' (t) \overset{\text{def}}{=} - \int_A \langle \nabla u_t, \nabla f_t \rangle \cdot d\text{vol} = \int_A u_t \cdot d\Delta f_t.
\]

Therefore

\[
w_u|_{t_0}^{t_1} = \int_{t_0}^{t_1} \int_A u_t \cdot d\Delta f_t.
\]

The last formula extends to an arbitrary Borel function \( u : A \to \mathbb{R} \) with bounded support. Applying it to the characteristic function of \( E \) we get the result.

**2 Games with Hamilton–Jacobi shifts.**

Let \( A \in \text{Alex}^m[0] \). For a function \( f : A \to \mathbb{R} \cup \{+\infty\} \), let us define its Hamilton–Jacobi shift\footnote{There is a lot of similarity between the Hamilton–Jacobi shift of a function and an equidistant for a hypersurface.} \( \mathcal{H}_t f : A \to \mathbb{R} \) for time \( t > 0 \) as follows

\[
(\mathcal{H}_t f) (x) \overset{\text{def}}{=} \inf_{y \in A} \left\{ f(y) + \frac{1}{2t} |xy|^2 \right\}.
\]

We say that \( \mathcal{H}_t f \) is well defined if the above infimum is \( > -\infty \) everywhere in \( A \). Clearly,

\[
\mathcal{H}_{t_0+t_1} f = \mathcal{H}_{t_1} \mathcal{H}_{t_0} f,
\]

for any \( t_0, t_1 > 0 \).
Note that for \( t > 0 \), \( f_t = \mathcal{H}_t f \) forms a \( \frac{1}{t} \)-concave family, thus, we can apply 1.2 and 1.3. The next theorem gives a more delicate property of the gradient flow for such families; it is an analog of [Petrunin 07, 3.3.6].

2.1. Claim. Let \( A \in \text{Alex}^m[0], f_0: A \to \mathbb{R} \) be function and let \( f_t = \mathcal{H}_t f_0 \) be well defined for \( t \in (0, 1) \). Assume \( \gamma: [0, 1] \to A \) is a geodesic path which is an \( f_t \)-gradient curve for \( t \in (0, 1) \) and \( \alpha: (0, 1) \to A \) is another \( f_t \)-gradient curve. Then if for some \( t_0 \in (0, 1) \), \( \alpha(t_0) = \gamma(t_0) \) then \( \alpha(t) = \gamma(t) \) for all \( t \in (0, 1) \).

Proof. Note that function \( \ell = \ell(t) = |\alpha(t)\gamma(t)| \) is locally Lipschitz in \((0,1) \), according to 1.2, it is sufficient to show that

\[
\ell' \geq -[\frac{1}{t} + \frac{2}{1-t}] \ell
\]

for almost all \( t \).

Since \( \alpha \) is locally Lipschitz, for almost all \( t \), \( \alpha^+(t) \) and \( \alpha^-(t) \) are well defined and opposite\(^3\) to each other.

Fix such \( t \) and set \( x = \gamma(0), z = \gamma(t), y = \gamma(1), p = \alpha(t) \), so \( \ell(t) = |pz| \).

Note that function

\[
f_t + \frac{1}{2(1-t)} \text{dist}^2_y
\]

has a minimum at \( z \). Extend a geodesic \([zp]\) by a both-sides infinite unit-speed quasigeodesic\(^4\) \( \sigma: \mathbb{R} \to A \), so \( \sigma(0) = z \) and \( \sigma^+(0) = \gamma(z) \). The function \( f_t \circ \sigma: \mathbb{R} \to \mathbb{R} \) is \( \frac{1}{t} \)-concave and from \( 3 \),

\[
f_t \circ \sigma(s) \geq f_t(z) + \langle \gamma^+(t), [zp] \rangle \cdot s - \frac{1}{2(1-t)} s^2.
\]

It follows that

\[
\langle \nabla_p f_t, \sigma^+(t) \rangle \geq d_p f_t(\sigma^+(t)) = (f_t \circ \sigma)^+ (t) \geq \langle \gamma^+(t), [zp] \rangle \geq \langle \frac{1}{t} + \frac{2}{1-t} \rangle \ell.
\]

Now,

1. Vectors \( \sigma^\pm(t) \) are polar, thus \( \langle \alpha^\pm(t), \sigma^+(t) \rangle + \langle \alpha^\pm(t), \sigma^-(t) \rangle \geq 0 \).
2. Vectors \( \alpha^\pm(t) \) are opposite, thus \( \langle \alpha^+(t), \sigma^+(t) \rangle + \langle \alpha^-(t), \sigma^-(t) \rangle = 0 \).
3. \( \alpha^+(t) = \nabla_p f_t \) and \( \sigma^-(t) = \gamma(z) \)

Thus, \( \langle \nabla_p f_t, \sigma^+(t) \rangle + \langle \alpha^+(t), [zp] \rangle = 0 \). Therefore

\[
\ell' = -\langle \alpha^+(t), [zp] \rangle - \langle \gamma^+(t), [zp] \rangle \geq -[\frac{1}{t} + \frac{2}{1-t}] \ell.
\]

2.2. Proposition. Let \( A \in \text{Alex}^m[0], f: A \to \mathbb{R} \) be a bounded continuous function and let \( f_t = \mathcal{H}_t f \). Assume \( \gamma: (0, a) \to A \) is an \( f_t \)-gradient curve which is also a constant-speed geodesic. Assume that function

\[
h(t) \overset{\text{def}}{=} \text{Trace Hess}_{\gamma(t)} f_t
\]

\(^3\)i.e. \( |\alpha^+(t)| = |\alpha^-(t)| \) and \( \angle(\alpha^+(t), \alpha^-(t)) = \pi \)

\(^4\)A careful proof of existence of quasigeodesics can be found in [Petrunin 07].
is defined for almost all \( t \in (0, a) \). Then

\[ h' \leq -\frac{1}{m} h^2 \]

in the sense of distributions; i.e. for any non-negative Lipschitz function \( u: (0, a) \to \mathbb{R} \) with compact support

\[ \int_0^a (\frac{1}{m} h^2 u - h u') \, dt \geq 0. \]

**Proof.** Since \( h \) is defined a.e., all \( T_{\gamma(t)} \) for \( t \in (0, a) \) are isometric to Euclidean \( m \)-space. From 2, \( f_{t_1}(x) = \inf_{y \in A} \left\{ f_{t_0}(y) + \frac{|xy|^2}{2(t_1 - t_0)} \right\}. \)

Thus, for a parallel transportation \( \tau: T_{\gamma(t_0)} \to T_{\gamma(t_1)} \) along \( \gamma \), we have

\[ \text{Hess}_{\gamma(t_1)} f_{t_1}(y) \leq \text{Hess}_{\gamma(t_0)} f_{t_0}(x) + \frac{|\tau(t) y|^2}{2(t_1 - t_0)} \]

for any \( x \in T_{\gamma(t_0)} \) and \( y \in T_{\gamma(t_1)} \). Taking trace leads to the result. \( \square \)

### 3 Proof of the main theorem

Let \( A \in \text{Alex}^m[0] \); in particular \( A \) is a proper geodesic space. Let \( \mu_t \) be a family of probability measures on \( A \) for \( t \in [0, 1] \) which forms a geodesic path\(^5\) in \( P_2 A \) and both \( \mu_0 \) and \( \mu_1 \) are absolutely continuous with respect to volume on \( A \).

It is sufficient\(^6\) to show that the function

\[ \Theta: t \mapsto U_m \mu_t \]

is concave.

According to [Villani, 7.22], there is a probability measure \( \pi \) on the space of all geodesic paths in \( A \) which satisfies the following: If \( \Gamma = \text{supp} \pi \) and \( e_\tau: \Gamma \to A \) is evaluation map \( e_\tau: \gamma \mapsto \gamma(t) \) then \( \mu_t = e_\tau# \pi \).

The measure \( \pi \) is called the *dynamical optimal coupling* for \( \mu_t \) and the measure \( \pi = (e_0, e_1)#\pi \) is the corresponding *optimal transference plan*. The space \( \Gamma \) will be considered further equipped with the metric \( \| \gamma \gamma' \| = \max_{t \in [0, 1]} \| \gamma(t) \gamma'(t) \| \).

**First we present \( \mu_t \) as the push-forward for gradient flows of two opposite families of functions.** According to [Villani, 5.10], there are optimal price functions \( \varphi, \psi: A \to \mathbb{R} \) such that

\[ \varphi(y) - \psi(x) \leq \frac{1}{2} |xy|^2 \]

for any \( x, y \in A \) and equality holds for any \( (x, y) \in \text{supp} \pi \). We can assume that \( \psi(x) = +\infty \) for \( x \notin \text{supp} \mu_0 \) and \( \varphi(y) = -\infty \) for \( y \notin \text{supp} \mu_1 \).

\(^5\)i.e. constant-speed minimizing geodesic defined on \([0, 1]\)

\(^6\)It follows from [Villani, 30.32] since Alexandrov’s spaces are nonbranching.
Consider two families of functions
\[ \psi_t = \mathcal{H}_t \psi \quad \text{and} \quad \varphi_t = \mathcal{H}_{1-t}(-\varphi). \]
Clearly, \( \psi_t \) forms a \( \frac{1}{t} \)-concave family for \( t \in (0, 1] \) and \( \varphi_t \) forms a \( \frac{1}{1-t} \)-concave family for \( t \in [0, 1). \)

It is straightforward to check that for any \( \gamma \in \Gamma \) and \( t \in (0, 1) \)
\[ \pm \langle \gamma^\pm(t), v \rangle = d_{\gamma(t)} \psi_t(v) = -\varphi_{\gamma(t)}(v); \]
in particular,
\[ \gamma^+(t) = \nabla \psi_t \quad \text{and} \quad \gamma^-(t) = \nabla \varphi_t. \]

For \( 0 < t_0 \leq t_1 \leq 1 \), let us consider the maps \( \Psi_{t_0, t_1}: A \to A \) — the gradient flow of \( \psi_t \), defined by
\[ \Psi_{t_0, t_1}(t_0) = \alpha(t_1) \quad \text{if} \quad \alpha^+(t) = \nabla \psi_t. \]
Similarly, \( 0 \leq t_0 \leq t_1 < 1 \), define map \( \Phi_{t_1, t_0}: A \to A \)
\[ \Phi_{t_1, t_0}(t_1) = \beta(t_0) \quad \text{if} \quad \beta^-(t) = \nabla \varphi_t. \]

According to 1.2,
\[ \Psi_{t_0, t_1} \text{ is } \frac{t_1-t_0}{t_0} \text{-Lipschitz} \quad \text{and} \quad \Phi_{t_1, t_0} \text{ is } \frac{t_1-t_0}{1-t_0} \text{-Lipschitz}. \]

From \( \Phi \), \( e_{t_1} = \Psi_{t_0, t_1} \circ e_{t_0} \) and \( e_{t_0} = \Phi_{t_1, t_0} \circ e_{t_1} \). Thus, for any \( t \in (0, 1) \), the map \( e_t: \Gamma \to A \) is bi-Lipschitz. In particular, for any measure \( \chi \) on \( A \), there is a uniquely determined one-parameter family of “pull-back” measures \( \chi_t^{\ast} \) on \( \Gamma \), i.e. such that \( \chi_t^{\ast} E = \chi(e_t E) \) for any Borel subset \( E \subset \Gamma \).

Fix some \( z_0 \in (0, 1) \) (one can take \( z_0 = \frac{1}{4} \)) and equip \( \Gamma \) with the measure \( \nu = \text{vol}^{z_0}_\ast \). Thus, from now on “almost everywhere” has sense in \( \Gamma \), \( \Gamma \times (0, 1) \) and so on.

**Now we will represent \( \Theta \) in terms of families of functions on \( \Gamma \).** Note that \( \mu_t = \Psi_{t_1} \ast \mu_1 \) and \( \Psi_{t_1} \) is \( \frac{1}{t_1-t_0} \)-Lipschitz. Since \( \mu_1 \) is absolutely continuous, so is \( \mu_t \) for all \( t \). Set \( \mu_t = \rho_t \cdot \text{vol} \). Note that from \( \Phi \), we get that
\[
\left( \frac{1-t_1}{1-t_0} \right)^m \leq \frac{\rho_t(\gamma(t_1))}{\rho_t(\gamma(t_0))} \leq \left( \frac{t_1}{t_0} \right)^m
\]
for almost all \( \gamma \in \Gamma \) and \( 0 < t_0 < t_1 < 1 \). For \( \gamma \in \Gamma \) set \( r_t(\gamma) = \rho_t(\gamma(t)) \). Then
\[
\Theta(t) = \int_A \rho_t^{-\frac{m}{n}} \cdot d\mu_t = \int_\Gamma r_t^{-\frac{m}{n}} \cdot d\nu.
\]

In particular, \( \Theta \) is locally Lipschitz in \( (0, 1) \).

**Next we show that the measure \( \Delta \varphi_t \) is absolutely continuous on \( e_{t_1} \Gamma \) and that \( r_t(\gamma(t)) = \rho_t(\gamma(t)) \cdot \Delta \varphi_t \) in some weak sense.** From \( \Phi \), \( \text{vol}_t^{z_0} = e^{w_1 \ast \nu} \) for some Borel function \( w_1: \Gamma \to \mathbb{R} \). Thus
\[
\text{vol}_{e_t} E = \int_E e^{w_1} \cdot d\nu
\]
\textsuperscript{7}Note that usually \( \varphi_t \) is defined with opposite sign, but I wanted to work with semiconcave functions only.
for any Borel subset \( E \subset \Gamma \). Moreover, for almost all \( \gamma \in \Gamma \), we have that function \( t \mapsto w_t(\gamma) \) is locally Lipschitz in \((0, 1)\) (more precisely, \( t \mapsto w_t(\gamma) \) coincides with a Lipschitz function outside of a set of zero measure). In particular \( \frac{\partial w_t}{\partial t} \) is well defined a.e. in \( \Gamma \times (0, 1) \) and moreover

\[
w_t = \int_0^t \frac{\partial w_t}{\partial t} \, dt.
\]

Further, from 2.1, if \( 0 < t_0 \leq t_1 < 1 \) then for any \( \gamma \in \Gamma \),

\[
\Psi_{t_0, t_1}(x) = \gamma(t_1) \iff x = \gamma(t_0),
\]

\[
\Phi_{t_1, t_0}(x) = \gamma(t_0) \iff x = \gamma(t_1).
\]

Thus, for any Borel subset \( E \subset \Gamma \),

\[
e_{t_1} E = \Psi_{t_0, t_1} \circ e_{t_0} E = \Phi_{t_0, t_1}^{-1}(e_{t_0} E),
\]

\[
e_{t_0} E = \Phi_{t_1, t_0} \circ e_{t_1} E = \Psi_{t_0, t_1}^{-1}(e_{t_1} E)
\]

Set

\[
v(t) \overset{\text{def}}{=} \text{vol} e_{t} E = \int_E e^{w_t} \, dv.
\]

From 1.3,

\[
v'(t) \overset{\text{def}}{=} \partial \Psi_t(e_t E) \overset{\text{def}}{=} -\partial \phi_t(e_t E).
\]

Thus, \( \Delta \psi_t + \Delta \phi_t = 0 \) everywhere on \( e_t \Gamma \). From 1.1,

\[
\Delta \psi_t \leq \frac{m}{t} \cdot \text{vol}, \quad \Delta \phi_t \leq \frac{m}{1-t} \cdot \text{vol}.
\]

Thus, both restrictions \( \Delta \psi_t|_{e_t \Gamma} \) and \( \Delta \phi_t|_{e_t \Gamma} \) are absolutely continuous with respect to volume. Therefore

\[
v'(t) \overset{\text{def}}{=} \int_{e_t E} \text{Trace Hess} \phi_t \cdot d \text{vol}.
\]

For the one parameter family of functions \( h_t(\gamma) = \text{Trace Hess}_{\gamma(t)} \phi_t \), we have

\[
v^t_{x_0} = \int_E (e^{w_t} - 1) \, dv = \int_{x_0}^t d \xi \cdot \int_E h_t e^{w_t} \, dv
\]

or any Borel set \( E \subset \Gamma \). Equivalently,

\[
\frac{\partial w_t}{\partial t} \overset{\text{def}}{=} h_t
\]

From 2.2,

\[
\frac{\partial h_t}{\partial t} \leq \frac{1}{m} h_t^2
\]
Thus, for almost all $\gamma \in \Gamma$, the following inequality holds in the sense of distributions:

$$\frac{\partial^2}{\partial t^2} \exp\left(\frac{w_t(\gamma)}{m}\right) = \left(\frac{1}{m^2} + \frac{1}{m} \frac{\partial h_t}{\partial t}\right) \exp\left(\frac{w_t(\gamma)}{m}\right) \leq 0;$$

i.e. $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ is concave — more precisely, $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ coincides with a concave function almost everywhere.

Clearly, for any $t$ we have $\mu = r_t e^{w_t} \nu$. Thus, for almost all $\gamma$ there is a non-negative Borel function $a : \Gamma \rightarrow \mathbb{R}$ such that $r_t \equiv a e^{-w_t}$. Continue $\Theta$,

$$\Theta(t) = \int_\Gamma r_t^{-\frac{1}{m}} \cdot d\nu = \int_\Gamma e^{\frac{w_t}{m}} \cdot \sqrt{a} \cdot d\nu$$

i.e. $\Theta$ is concave as an average of concave functions. Again, more precisely, $\Theta$ coincides with a concave function a.e., but since $\Theta$ is locally Lipschitz in $(0, 1)$ we get that $\Theta$ is concave.

References

[Petrunin 07] Petrunin, A. Semiconcave Functions in Alexandrov’s Geometry, Surveys in Differential Geometry XI.