CHAPTER 1  Linear Equations

possible (in absolute value). This swap is performed even if the initial cursor entry is nonzero, as long as there is an entry below with a larger absolute value. In the first example worked in the text, we would start by swapping the first row with the last:

\[
\begin{bmatrix}
1 & 2 & 3 & 39 \\
1 & 3 & 2 & 34 \\
3 & 2 & 1 & 26
\end{bmatrix}
\rightarrow
\begin{bmatrix}
3 & 2 & 1 & 26 \\
1 & 3 & 2 & 34 \\
1 & 2 & 3 & 39
\end{bmatrix}
\]

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix \( A \) into a matrix \( B \) by a sequence of elementary row operations and if \( B \) is in reduced row-echelon form, is it necessarily true that \( B = \text{ref}(A) \)? Fortunately (and perhaps surprisingly) this is indeed the case.

In this text, we will not utilize this fact, so there is no need to present the somewhat technical proof. If you feel ambitious, try to work out the proof yourself after studying Chapter 3. (See Exercises 3.3.64 through 3.3.67.)

EXERCISES 1.2

GOAL Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work. Solve the system in Exercise 8 for the variables \( x_1, x_2, x_3, x_4, \) and \( x_5 \).

1. \[
\begin{align*}
x + y - 2z &= 5 \\
2x + 3y + 4z &= 2
\end{align*}
\]

2. \[
\begin{align*}
3x + 4y - z &= 8 \\
6x + 8y - 2z &= 3
\end{align*}
\]

3. \[
\begin{align*}
3y + 4z &= 4 \\
x + y &= 1
\end{align*}
\]

4. \[
\begin{align*}
x + y &= 0 \\
2x - y &= 5
\end{align*}
\]

5. \[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 0 \\
x_1 + x_2 &= 0 \\
x_1 + x_4 &= 0
\end{align*}
\]

6. \[
\begin{align*}
x_1 - 7x_2 + x_3 + 3x_5 &= 3 \\
x_3 - 2x_5 &= 2 \\
x_4 + x_5 &= 1
\end{align*}
\]

7. \[
\begin{align*}
x_1 + 2x_2 + 2x_4 + 3x_5 &= 0 \\
x_3 + 3x_2 + 2x_5 &= 0 \\
x_3 + 4x_4 - x_5 &= 0 \\
x_5 &= 0
\end{align*}
\]

8. \[
\begin{align*}
x_2 + 2x_4 + 3x_5 &= 0 \\
4x_4 + 8x_5 &= 0
\end{align*}
\]

9. \[
\begin{align*}
x_1 + 2x_2 + x_3 + x_4 &= 2 \\
x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 0 \\
x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2
\end{align*}
\]

10. \[
\begin{align*}
x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\
5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\
-2x_1 - 2x_2 + x_3 + 2x_4 &= -3 \\
11x_1 + 6x_2 + 4x_3 + x_4 &= 11
\end{align*}
\]

11. \[
\begin{align*}
x_1 + 2x_3 + 4x_4 &= -8 \\
x_2 - 3x_3 - x_4 &= 6 \\
3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0 \\
x_2 + 3x_3 + 4x_4 &= -12
\end{align*}
\]

12. \[
\begin{align*}
2x_1 - 3x_3 &+ 7x_5 + 7x_6 = 0 \\
-2x_1 + x_2 + 6x_5 - 6x_6 &= 0 \\
x_2 - 3x_3 + x_5 + 5x_6 &= 0 \\
-x_2 + x_3 + x_5 + x_6 &= 0
\end{align*}
\]

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. \[
\begin{align*}
x_1 + 11y + 19z &= 0 \\
7x + 23y + 39z &= 10 \\
-4x - 3y - 2z &= 6
\end{align*}
\]

14. \[
\begin{align*}
3x + 6y + 14z &= 22 \\
7x + 14y + 30z &= 46 \\
4x + 8y + 7z &= 6
\end{align*}
\]

15. \[
\begin{align*}
3x + 5y + 3z &= 25 \\
7x + 9y + 19z &= 65 \\
-4x + 5y + 11z &= 5
\end{align*}
\]

16. \[
\begin{align*}
3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 &= 53 \\
7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 &= 105 \\
-4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 &= 11
\end{align*}
\]

17. \[
\begin{align*}
2x_1 - 4x_2 + 3x_3 + 5x_4 + 6x_5 &= 37 \\
4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 &= 74 \\
-2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 &= 20 \\
x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 &= 26 \\
x_1 - 10x_2 + 4x_3 + 6x_4 + 3x_5 &= 24
\end{align*}
\]

18. Determine which of the matrices below are in reduced row-echelon form:

a. \[
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]


1.2 Matrices, Vectors, and Gauss–Jordan Elimination

30. Find the polynomial of degree 3 [a polynomial of the form \( f(t) = a + bt + ct^2 + dt^3 \)] whose graph goes through the points \((0, 1), (1, 0), (-1, 0),\) and \((2, -15)\). Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points \((1, 1), (2, -1), (3, -59), (-1, 5),\) and \((-2, -29)\). Graph this polynomial.

32. Cubic Splines. Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points \((a_i, b_i)\) are given to you, and your job is to connect the dots in a reasonably smooth way. Let \(a_{i+1} > a_i\).

One method often employed in such design problems is the technique of cubic splines. We choose \(f_i(t)\), a polynomial of degree \(\leq 3\), to define the shape of the ride between \((a_{i-1}, b_{i-1})\) and \((a_i, b_i)\), for \(i = 1, \ldots, n\).

Obviously, it is required that \(f_i(a_i) = b_i\) and \(f_i(a_{i-1}) = b_{i-1}\), for \(i = 1, \ldots, n\). To guarantee a smooth ride at the points \((a_i, b_i)\), we want the first and the second derivatives of \(f_i\) and \(f_{i+1}\) to agree at these points:

\[
\begin{align*}
&f'_i(a_i) = f'_{i+1}(a_i) \\
&f''_i(a_i) = f''_{i+1}(a_i), \quad \text{for } i = 1, \ldots, n - 1.
\end{align*}
\]

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

\[
f'_i(a_0) = f'_i(a_n) = 0.
\]

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (Note: It can be shown that this system has a unique solution.)
33. Find the polynomial \( f(t) \) of degree 3 such that \( f(1) = 1, \ f(2) = 5, \ f'(1) = 2, \) and \( f'(2) = 9, \) where \( f'(t) \) is the derivative of \( f(t) \). Graph this polynomial.

34. The dot product of two vectors

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}
\]

in \( \mathbb{R}^n \) is defined by

\[
\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.
\]

Note that the dot product of two vectors is a scalar. We say that the vectors \( \vec{x} \) and \( \vec{y} \) are perpendicular if \( \vec{x} \cdot \vec{y} = 0 \).

Find all vectors in \( \mathbb{R}^3 \) perpendicular to

\[
\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.
\]

Draw a sketch.

35. Find all vectors in \( \mathbb{R}^4 \) that are perpendicular to the three vectors

\[
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix},
\]

(See Exercise 34.)

36. Find all solutions \( x_1, \ x_2, \ x_3 \) of the equation

\[
\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,
\]

where

\[
\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 7 \\ 5 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 6 \\ 9 \end{bmatrix}.
\]

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, \( I_1, I_2, I_3 \). What outputs \( x_1, x_2, x_3 \) should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.

38. If we consider more than three industries in an input-output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have the industries \( I_1, I_2, \ldots, I_n \), with outputs \( x_1, x_2, \ldots, x_n \). The output vector is

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.
\]

The consumer demand vector is

\[
\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},
\]

where \( b_i \) is the consumer demand on industry \( I_i \). The demand vector for industry \( I_j \) is

\[
\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},
\]

where \( a_{ij} \) is the demand industry \( I_i \) puts on industry \( I_j \), for each $1 of output industry \( I_j \) produces. For example, \( a_{12} = 0.5 \) means that industry \( I_1 \) needs 50¢ worth of products from industry \( I_3 \) for each $1 worth of goods \( I_2 \) produces. The coefficient \( a_{ij} \) need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of \( x_j\vec{v}_j \)?

c. What is the meaning in economic terms of \( x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} \)?

d. What is the meaning in economic terms of the equation

\[
x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}?
\]
39. Consider the economy of Israel in 1958. The three industries considered here are

\[ I_1 : \text{agriculture,} \]
\[ I_2 : \text{manufacturing,} \]
\[ I_3 : \text{energy.} \]

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time. We are told that

\[
\tilde{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \tilde{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},
\]
\[
\tilde{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \tilde{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.
\]

a. Why do the first components of \( \tilde{v}_2 \) and \( \tilde{v}_3 \) equal 0?

b. Find the outputs \( x_1, x_2, x_3 \) required to satisfy demand.

40. Consider some particles in the plane with position vectors \( \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n \) and masses \( m_1, m_2, \ldots, m_n \).

The position vector of the center of mass of this system is

\[
\tilde{r}_{cm} = \frac{1}{M} (m_1 \tilde{r}_1 + m_2 \tilde{r}_2 + \cdots + m_n \tilde{r}_n),
\]

where \( M = m_1 + m_2 + \cdots + m_n \).

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that the plate can be supported at the point \( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \); that is, \( \tilde{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \)? Assume that the mass of the plate itself is negligible.

41. The momentum \( \tilde{P} \) of a system of \( n \) particles in space with masses \( m_1, m_2, \ldots, m_n \) and velocities \( \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \) is defined as

\[
\tilde{P} = m_1 \tilde{v}_1 + m_2 \tilde{v}_2 + \cdots + m_n \tilde{v}_n.
\]

Now consider two elementary particles with velocities

\[
\tilde{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \tilde{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.
\]

The particles collide. After the collision, their respective velocities are observed to be

\[
\tilde{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \tilde{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.
\]

Assume that the momentum of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? (See the accompanying figure.)

42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during...
this hour were exactly the same as those entering it.

What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

43. Let $S(t)$ be the length of the $r$th day of the year in Mumbai (formerly known as Bombay), India (measured in hours, from sunrise to sunset). We are given the following values of $S(t)$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$S(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>11.5</td>
</tr>
<tr>
<td>74</td>
<td>12</td>
</tr>
<tr>
<td>273</td>
<td>12</td>
</tr>
</tbody>
</table>

For example, $S(47) = 11.5$ means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function $S(t)$ is well approximated by a trigonometric function of the form

$$S(t) = a + b \cos \left( \frac{2\pi t}{365} \right) + c \sin \left( \frac{2\pi t}{365} \right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

44. Kyle is getting some flowers for Kate, his Valentine. Being of a precise analytical mind, he plans to spend exactly $24 on a bunch of exactly two dozen flowers. At the flower market they have lilies ($$3 each), roses ($2 each), and daisies ($0.50 each). Kyle knows that Kate loves lilies; what is he to do?

45. Consider the equations

$$\begin{align*}
x + 2y + 3z &= 4, \\
x + ky + 4z &= 6, \\
x + 2y + (k + 2)z &= 6, \end{align*}$$

where $k$ is an arbitrary constant.

(a) For which values of the constant $k$ does this system have a unique solution?

(b) When is there no solution?

(c) When are there infinitely many solutions?

46. Consider the equations

$$\begin{align*}
y + 2kz &= 0, \\
x + 2y + 6z &= 2, \\
kx + 2z &= 1, \end{align*}$$

where $k$ is an arbitrary constant.

(a) For which values of the constant $k$ does this system have a unique solution?

(b) When is there no solution?

(c) When are there infinitely many solutions?

47. Find all solutions $x_1, x_2, x_3, x_4, x_5, x_6$ of the system $x_2 = \frac{1}{2}(x_1 + x_3), x_3 = \frac{1}{2}(x_2 + x_4), x_4 = \frac{1}{2}(x_3 + x_5), x_5 = \frac{1}{2}(x_4 + x_6), x_6 = \frac{1}{2}(x_5 + x_1)$.

48. For an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, x_3, \ldots, x_n$ of the simultaneous equations $x_1 = \frac{1}{2}(x_1 + x_3), x_3 = \frac{1}{2}(x_2 + x_4), \ldots, x_n = \frac{1}{2}(x_{n-2} + x_n)$.

Note that we are asked to solve the simultaneous equations $x_1 = \frac{1}{2}(x_{n-1} + x_n)$, for $k = 2, 3, \ldots, n - 1$.

49. Consider the system

$$\begin{align*}
2x + y &= C, \\
3y + z &= C, \\
x + 4z &= C, \end{align*}$$

where $C$ is a constant. Find the smallest positive integer $C$ such that $x, y,$ and $z$ are all integers.

50. Find all the polynomials $f(t)$ of degree $\leq 3$ such that $f(0) = 3, f(1) = 2, f(2) = 0$, and $\int_0^1 f(t) dt = 4$. (If you have studied Simpson’s rule in calculus, explain the result.)


52. Students are buying books for the new semester. Brigitte buys the German grammar book and the German novel, Die Leiden des jungen Werther, for $€64 in total. Claude spends $€98 on the linear algebra text and the German grammar book. While Denise buys the linear algebra text and Werther, for $€76. How much does each of the three books cost?
53. At the beginning of a political science class at a large university, the students were asked which term, liberal or conservative, best described their political views. They were asked the same question at the end of the course, to see what effect the class discussions had on their views. Of those that characterized themselves as “liberal” initially, 30% held conservative views at the end. Of those who were conservative initially, 40% moved to the liberal camp. It turned out that there were just as many students with conservative views at the end as there had been liberal students at the beginning. Out of the 260 students in the class, how many held liberal and conservative views at the beginning of the course and at the end? (No students joined or dropped the class between the surveys, and they all participated in both surveys.)

54. At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in Section A switch to Section B in the first few weeks of class, while 30% of the students in Section B switch to A, resulting in a net loss of 4 students for Section B. How large were the two sections at the beginning of the semester? No students dropped Linear Algebra (why would they?) or joined the course late.

Historical Problems

55. Five cows and two sheep together cost ten liang\textsuperscript{14} of silver. Two cows and five sheep together cost eight liang of silver. What is the cost of a cow and a sheep, respectively? (Nine Chapters, Chapter 8, Problem 7)

56. If you sell two cows and five sheep and you buy 13 pigs, you gain 1,000 coins. If you sell three cows and three pigs and buy nine sheep, you break even. If you sell six sheep and eight pigs and you buy five cows, you lose 600 coins. What is the price of a cow, a sheep, and a pig, respectively? (Nine Chapters, Chapter 8, Problem 8)

57. You place five sparrows on one of the pans of a balance and six swallows on the other pan; it turns out that the sparrows are heavier. But if you exchange one sparrow and one swallow, the weights are exactly balanced. All the birds together weigh 1 jin. What is the weight of a sparrow and a swallow, respectively? [Give the answer in liang, with 1 jin = 16 liang]. (Nine Chapters, Chapter 8, Problem 9)

58. Consider the task of pulling a weight of 40 dan\textsuperscript{16} up a hill; we have one military horse, two ordinary horses, and three weak horses at our disposal to get the job done. It turns out that the military horse and one of the ordinary horses, pulling together, are barely able to pull the weight (but they could not pull any more). Likewise, the two ordinary horses together with one weak horse are just able to do the job, as are the three weak horses together with the military horse. How much weight can each of the horses pull alone? (Nine Chapters, Chapter 8, Problem 12)

59. Five households share a deep well for their water supply. Each household owns a few ropes of a certain length, which varies only from household to household. The five households, A, B, C, D, and E, own 2, 3, 4, 5, and 6 ropes, respectively. Even when tying all their ropes together, none of the households alone is able to reach the water, but A’s two ropes together with one of B’s ropes just reach the water. Likewise, B’s three ropes with one of C’s ropes, C’s four ropes with one of D’s ropes, D’s five ropes with one of E’s ropes, and E’s six ropes with one of A’s ropes all just reach the water. How long are the ropes of the various households, and how deep is the well? 
Commentary: As stated, this problem leads to a system of 5 linear equations in 6 variables; with the given information, there are no solutions. The Nine Chapters give one particular solution, where the depth of the well is 7 zhang,\textsuperscript{17} 2 chi, 1 cun, or 721 cun (since 1 zhang = 10 chi and 1 chi = 10 cun). Using this particular value for the depth of the well, find the lengths of the various ropes.

60. “A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?” (From the Mathematical Manual by Zhang Qiujuan, Chapter 3, Problem 38; 5th century A.D.)
Commentary: This famous Hundred Fowl Problem has reappeared in countless variations in Indian, Arabic, and European texts (see Exercises 61 through 64); it has remained popular to this day (see Exercise 44 of this section).

61. “Pigeons are sold at the rate of 5 for 3 panas, sarasabirds at the rate of 7 for 5 panas, swans at the rate of 9 for 7 panas, and peacocks at the rate of 3 for 9 panas. A man was told to bring 100 birds for 100 panas for the amusement of the King’s son. What does he pay for each of the various kinds of birds that he buys?” (From the Ganita-Sara Sangraha by Mahavira, India; 9th century A.D.) Find one solution to this problem.

62. “A duck costs four coins, five sparrows cost one coin, and a rooster costs one coin. Somebody buys 100 birds for 100 coins. How many birds of each kind can he buy?” (From the Key to Arithmetic by Al-Kashi; 15th century)

\textsuperscript{14} A liang was about 16 grams at the time of the Han Dynasty.

\textsuperscript{15} See page 1; we present some of the problems from the Nine Chapters on the Mathematical Art in a free translation, with some additional explanations, since the scenarios discussed in a few of these problems are rather unfamiliar to the modern reader.

\textsuperscript{16} 1 dan = 120 jin = 1920 liang. Thus a dan was about 30 kg at that time.

\textsuperscript{17} 1 zhang was about 2.3 meters at that time.
63. "A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him \( \frac{1}{2} \) a crown a-piece; the goats, 1 \( \frac{1}{2} \) crown; and the hogs 3 \( \frac{1}{2} \) crowns. How many had he of each?" (From the Elements of Algebra by Leonhard Euler, 1770)

64. "A gentleman has a household of 100 persons and orders that they be given 100 measures of grain. He directs that each man should receive three measures, each woman two measures, and each child half a measure. How many men, women, and children are there in this household?" We are told that there is at least one man, one woman, and one child. (From the Problems for Quickening a Young Mind by Alcuin [c.732–804], the Abbot of St. Martins at Tours. Alcuin was a friend and tutor to Charlemagne and his family at Aachen.)

65. A father, when dying, gave to his sons 30 barrels, of which 10 were full of wine, 10 were half full, and the last 10 were empty. Divide the wine and flasks so that there will be equal division among the three sons of both wine and barrels. Find all the solutions of this problem. (From Alcuin)

66. "Make me a crown weighing 60 minae, mixing gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must put in." (From the Greek Anthology by Metrodorus, 6th century A.D.)

67. Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together." "Give me the purse and I shall have three times as much as the two of you together" said the second merchant. The third merchant said "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together." If there are 60 coins (of equal value) in the purse, how much money does each merchant have? (From Mahavira)

68. 3 cows graze 1 field bare in 2 days, 7 cows graze 4 fields bare in 4 days, and 3 cows graze 2 fields bare in 5 days. It is assumed that each field initially provides the same amount, \( x \), of grass; that the daily growth, \( y \), of the fields remains constant; and that all the cows eat the same amount, \( z \), each day. (Quantities \( x \), \( y \), and \( z \) are measured by weight.) Find all the solutions of this problem. (This is a special case of a problem discussed by Isaac Newton in his Arithmetica Universalis, 1707.)

1.3 ON THE SOLUTIONS OF LINEAR SYSTEMS; MATRIX ALGEBRA

In this final section of Chapter 1, we will discuss two rather unrelated topics:

- First, we will examine how many solutions a system of linear equations can possibly have.
- Then, we will present some definitions and rules of matrix algebra.

The Number of Solutions of a Linear System

The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?

\[
a. \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hspace{1cm} b. \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hspace{1cm} c. \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}
\]

Solution

a. The third row represents the equation \( 0 = 1 \), so that there are no solutions. We say that this system is *inconsistent*.

b. The given augmented matrix represents the system

\[
\begin{align*}
x_1 + 2x_2 &= 1 \\
x_3 &= 2
\end{align*}
\]

or,

\[
\begin{align*}
x_1 &= 1 - 2x_2 \\
x_3 &= 2
\end{align*}
\]

We can assign an arbitrary value, \( t \), to the free variable \( x_2 \), so that the system has