Having defined the complex exponential function and having seen that it satisfies the same differentiation rule as the real exponential function:
\[
\frac{d}{dt} e^{\gamma t} = \gamma e^{\gamma t}
\]
enables us to immediately solve those linear homogeneous constant coefficient whose characteristic polynomial has complex roots.
So we consider a linear constant coefficient homogeneous second order ODE which we write as
\[
ay'' + by' + cy = 0
\]
We assume that the characteristic polynomial \(ar^2 + br + c\) has complex roots \(\gamma\) and \(\bar{\gamma}\). Also in this case a fundamental pair \(y_1, y_2\) can be obtained by taking real and imaginary parts of of \(e^{\gamma t}\). This is true for the same reason that it was true when \(\gamma\) was real.
Here, since
\[
\gamma = \alpha + i\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]
we see that \(L[e^{\alpha t} \cos \beta t + ie^{\alpha t} \sin \beta t] = 0\). And then by the superposition principle
\[
0 = L[e^{\alpha t} \cos \beta t + ie^{\alpha t} \sin \beta t] = L[e^{\alpha t} \cos \beta t] + iL[e^{\alpha t} \sin \beta t]
\]
Since the entire linear combination in the above formula is zero, both the real and imaginary parts individually are zero. This says that if \(y_1\) denotes \(e^{\alpha t} \cos \beta t\) and \(y_2\) denotes \(e^{\alpha t} \sin \beta t\), then each one has the property \(L[y_1] = 0\) and \(L[y_2] = 0\). (Although in the real case we could use \(y_1\) and \(y_2\) for either solution, here it is better to associate \(y_1\) with the solution have the cosine function in it.) That is, we have constructed two solutions to the ODE from the one complex-valued solution. Hopefully, they are not multiples of each other. That they are not multiples, can be seen by comparing their values at \(t = 0\) where \(y_2 = 0\) and \(y_1 \neq 0\) and at \(t = \frac{\pi}{2\beta}\) where \(y_1 = 0\) but \(y_2 \neq 0\).
We now illustrate the above ideas by solving
\[
y'' + 14y' + 74y = 0, \quad y(0) = 11, \quad y'(0) = -12
\]
The characteristic polynomial is \(r^2 + 14r + 74\) and we find its roots by completing the square (much more reliable than using the quadratic formula): \(r^2 + 14r + 72 = -74 + 72^2\), i.e., \((r + 7)^2 = -25\). Therefore the roots are \(\gamma = \pm i5\).
A complex solution is: \(e^{(-7+i5)t} = e^{-7t}(\cos(5t) + i \sin(5t))\). Taking \(y_1\) and \(y_2\) to be its real and imaginary parts we obtain
\[
y_1 = \text{Re}(e^{-7t}(\cos(5t) + i \sin(5t))) = e^{-7t} \cos 5t \quad y_2 = \text{Im}(e^{-7t}(\cos(5t) + i \sin(5t))) = e^{-7t} \sin 5t
\]
The general solution can be written more neatly as follows:
\[
y = e^{-7t}(c_1 \cos 5t + c_2 \sin 5t)
\]
Its derivative is:
\[
y' = -7y + 5e^{-7t}(-c_1 \sin 5t + c_2 \cos 5t)
\]
Plugging in the initial conditions gives
\[
y(0) = c_1 = 11, \quad y'(0) = -77 + 5c_2 = -12 \quad 5c_2 = 65 \quad c_2 = 13
\]
Therefore, the solution to the IVP is:
\[
y = e^{-7t}(11 \cos 5t + 13 \sin 5t)
\]
At this point we should make some observations about the long time behavior of solutions. The solutions in the complex case involve trig functions which are periodic (in fact, since the variable in the trig function is \(\beta t\), the period is \(2\pi/\beta\)) but the presence of the exponential with a negative coefficient for \(t\) makes its graph oscillate forever between two decaying exponentials. The only periodic feature of the trig functions that remains in the graph of these solutions is the time between consecutive crossing of the \(t\) axis which is always \(\pi/\beta\) (because there are two crossings of the \(t\) axis in each complete cycle) and there are infinitely many of them.
It is important to observe how the solution to this ODE changes as \(b\) or equivalently \(\alpha = -b/2a\) varies. If \(b\) is replaced by its negative, then the oscillations of the solution grow exponentially as \(t\) increases without bound. If
\( \beta \) is zero then the exponential function disappears from \( y_1 \) and \( y_2 \) and hence \( y = c_1 y_2 + c_2 y_2 \) is periodic with the same period \( 2\pi/\beta \). We will see later that such a combination of trig functions can be rewritten using a single cosine function that is shifted and multiplied by the amplitude of the oscillations.

Now let us summarize the long time behavior of the solutions of any 2nd order constant coefficient linear homogeneous ODE just by looking at the roots of its characteristic polynomial (except for the repeated roots case which we will deal with next time).

If the characteristic polynomial has complex roots we saw three different types of long time behavior are possible depending on the real part of the complex roots. If the real part is zero we have period solutions (oscillations with constant amplitude), if the real part is positive the solution oscillates forever between two decaying exponential functions and if the real part is negative the solution oscillates forever between two growing exponential functions. In all cases the intervals between crossings of the \( t \)-axis are of equal length and hence there are infinitely many of crossings.

If the characteristic polynomial has two positive roots then all solutions decay (approach zero) with increasing \( t \). If the characteristic polynomial has two positive negative roots then eventually all nonzero solutions increase or decrease without bound as \( t \) increases without bound, depending on the initial data (which determines \( c_1 \) and \( c_2 \)). Finally, if the characteristic polynomial has one positive and one negative root, then in addition to the above behavior a solution can also decay as \( t \) increases without bound.

The case of double roots will be dealt with tomorrow.

©2009 by Moses Glasner