Today we explain how building blocks can be found for PDE's. This method is called separation of variables. This method cannot be used to solve all PDE's but its claim to fame is that it does give a procedure for discovering building blocks for these three very important PDE's, as well as many others.

Suppose the unknown function \( u \) in our PDE is a function of the independent variables \( x \) and \( t \). The method is based on the assumption that the PDE has a solution which is expressible as a product of a function \( X(x) \) of only \( x \) and \( T(t) \) of only \( t \):

\[
\begin{align*}
    u(x, t) &= X(x)T(t) \\
    u_t &= u_x + u
\end{align*}
\]

such that \( u(x, 0) = 3e^{4x} \).

For this purpose let us assume that \( u(x, t) = X(t)T(t) \). Then plugging into the equation we obtain

\[
XT' = X'T + XT
\]

We now try to separate the variables (only functions of one variable appearing on each side):

\[
\frac{T'}{T} = \frac{X'}{X} = \lambda + 1
\]

Since a function of \( t \) is equal to a function of \( x \) in the above equation, both must be a constant; call it \( \lambda \). Thus we obtain two equations, each one very easy to solve:

\[
\begin{align*}
    T' &= \lambda \\
    X' &= \lambda - 1
\end{align*}
\]

Thus

\[
T = d_3e^{\lambda t} \\
X = d_1e^{(\lambda-1)x}
\]

and finally

\[
u(x, t) = d_3 \lambda e^{(t+\lambda)x} - x
\]

Now let us go back to the heat equation and see how one can discover the functions which we called "building blocks". That is we try to discover ALL the functions \( f(x) \) with \( 0 < x < L \) for which it is possible to find a NONZERO solution \( u(x, t) = X(x)T(t) \) for \( t > 0 \) satisfying the boundary conditions \( u(x, 0) = f(x) \) with \( 0 < x < L \) and \( u(0, t) = u(L, t) = 0 \) for any \( t > 0 \).

Assuming that \( u(x, t) = X(x)T(t) \) we rewrite the PDE as follows:

\[
X''(x)T(t) = X(x)T'(t)
\]

We more everything involving \( X \) and or \( x \) to the left hand side and everything involving \( T \) and or \( t \) to the right hand side:

\[
\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda
\]

We conclude that each side of the above formula is a constant, with respect to \( t \) and with respect to \( x \). We denote this constant by \(-\lambda\) (the minus sign is chosen in order to connect with the notation used last time and does not affect the mathematics here).
Now let us impose the boundary conditions
\[ u(0, t) = u(L, t) = 0, \quad t > 0, \quad u(x, 0) = f(x) \]
where \( f(x) \) is the initial temperature distribution function. The boundary value problem may be easy to solve for some \( f(x) \) but not for others. Our analysis should reveal for which \( f(x) \) it is easy to solve.

The first 2 conditions translate into the boundary value problem for \( X(x) \):
\[ X'' + \lambda X = 0 \quad X(0) = 0, \quad X(L) = 0 \]
But last time we found all eigenvalue eigenfunction pairs for this boundary value problem. They are
\[ \lambda = \left( \frac{n\pi}{L} \right)^2 \quad X(x) = \sin \left( \frac{n\pi}{L} \right) \]
where \( n \) is a positive integer. Solving the second equation \( T' = -\lambda T \) is even easier: \( T(t) = Ce^{-\lambda t} \). So if we choose \( X(x) = f(x) \) to be one of the above eigenfunctions, then the boundary condition \( u(x, 0) = f(x) \) requires that we choose \( C = 1 \) in the solution. We conclude that
\[ u(x, t) = X(x)T(t) = \sin \left( \frac{n\pi}{L} \right) e^{-\left( \frac{n\pi}{L} \right)^2 t} \]
What happens if we change the imposed boundary conditions to the following:
\[ u_x(0, t) = u_t(L, t) = 0, \quad t > 0, \quad u(x, 0) = f(x) \]
where \( f(x) \) is the initial temperature distribution function?

The first 2 conditions translate into the boundary value problem for \( X(x) \):
\[ X'' + \lambda X = 0 \quad X'(0) = 0, \quad X'(L) = 0 \]
In the last FAQ we found all eigenvalue eigenfunction pairs for this boundary value problem. They are
\[ \lambda = 0 \quad X(x) = 1 \]
and also
\[ \lambda = \left( \frac{n\pi}{L} \right)^2 \quad X(x) = \cos \left( \frac{n\pi}{L} x \right) \]
where \( n \) is a positive integer. Solving the second equation \( T' = -\lambda T \) is even easier: \( T(t) = Ce^{-\lambda t} \). So if we choose \( X(x) = f(x) \) to be one of the above eigenfunctions, then the boundary condition \( u(x, 0) = f(x) \) requires that we choose \( C = 1 \) in the solution. We conclude that
\[ u(x, t) = X(x)T(t) = 1 e^{\lambda t} = 1 \]
in the case \( \lambda = 0 \) is the eigenvalue and for the remaining eigenvalues:
\[ u(x, t) = X(x)T(t) = \cos \left( \frac{n\pi}{L} \right) e^{-\lambda t} = \cos \left( \frac{n\pi}{L} \right) e^{-\left( \frac{n\pi}{L} \right)^2 t} \]
We finish with the observation that not every PDE can be solved by the separation of variables. For example consider
\[ u_t + u_x = t \]
If we try to separate variables by setting \( u(x, t) = X(x)T(t) \) then we get
\[ XT'' + X'T = t \]
But no matter how hard the variables just will not separate.

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