Several times we encountered hints that the sequence of partial sums \( \{s_n(x)\} \) of the Fourier series of function \( f(x) \) on the interval \([-2, 2]\) does not converge to \( f(x) \). This is certainly puzzling. After all did we not derive the following formulas

\[
\begin{align*}
    a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) \, dx \\
    a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi}{L} x \right) \, dx \\
    b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi}{L} x \right) \, dx
\end{align*}
\]

for the Fourier coefficients under the assumption that the partial sums \( s_n(x) \) did converge to \( f(x) \)? That of course is what we did!

The puzzlement here stems from confusing a statement with its converse. We list both of them here in order to make the distinction:

**A.** If the sequence of partial sums \( \{s_n(x)\} \) of the Fourier series of a piecewise continuous function \( f(x) \) converge to \( f(x) \) on \([-L,l]\), then the Fourier coefficients, \( a_0 \), \( a_n \), and \( b_n \), are given by formulas (1), (2), and (3).

**B.** If a piecewise continuous \( f(x) \) is given on \([-L,L]\) and trigonometric polynomial \( s_n(x) \) are constructed according to (1) (2) and (3), then the sequence \( \{s_n(x)\} \) converges to \( f(x) \) on \([-L,L]\).

What we have verified is Statement A. But it is not logically equivalent to Statement B. and, in fact, as we suspected, Statement B. is not true without additional assumptions!

The Fourier Convergence Theorem tells us under what circumstances Statement B. is true.

i. \( f(x) \) and \( f'(x) \) are piecewise continuous on \((-\infty, \infty)\)

ii. \( f(x) \) is extended to be periodic with period \( 2L \) on \((-\infty, \infty)\)

iii. At points of discontinuity \( x_0 \) of \( f(x) \), \( f(x_0) \) is the average of the one-sided limits of \( f \) at \( x_0 \).

Under the above assumptions the Fourier Convergence Theorem states that:

\[
\lim_{n \to \infty} s_n(x) = f(x) \quad \text{for all} \quad x
\]

So for example if \( s_n(x) \) are the partial sums of the Fourier series of \( f(x) = xu(x) \) on the interval \([-2, 2]\), then we can find the following limits \( \lim_{n \to \infty} s_n(3/4) \) \( \lim_{n \to \infty} s_n(2) \) \( \lim_{n \to \infty} s_n(3) \) \( \lim_{n \to \infty} s_n(43) \) by insuring that \( f(x) \) satisfies the hypotheses, i., ii., and iii., of the Fourier Convergence Theorem.

Indeed \( f(3/4) = 3/4, f(2) = 1 \) is the average of the one-sided limits because \( x = 2 \) is a point of discontinuity, and \( f(3) = f(43) = 1 \), since \( f(x) \) has period \( 2L = 4 \).

There is another issue related to the convergence of \( s_n(x) \) to \( f(x) \). That is, if \( f \) has a discontinuity at a point \( x_0 \), then it is not possible for \( s_n(x) \) converge uniformly to \( f(x) \) near \( x_0 \). However, the Yale University mathematical physicist Josiah Willard Gibbs discovered rate of nonuniformity known as Gibbs’ phenomenon. Specifically, the discovery is that in any interval containing the discontinuity \( x_0 \), \( s_n(x_0) \) both overshoots and and undershoots \( f(x) \) by approximately 8% of the total jump at \( x_0 \).