Due Date: Thursday, Oct. 14, 2010

Write the final answer to the problems on this assignment and attach the worked out solutions!

Problem 1: Three balls are chosen randomly and with replacement from an urn containing 8 white, 4 black, and two orange balls. Suppose that we win $2 for each black ball and lose $1 for each white ball. Let $Y$ denote the number of black balls, $Z$ the number of white balls drawn, and let $X$ denote our winnings. What are the possible values of $X$, $Y$ and $Z$? Express $X$ as a function of $Y$ and $Z$.

Solution: $Y$ has values $0, 1, ..., 3$, $Z$ has values $0, 1, ..., 3$ and $X$ has values $-3, -2, -1, 0, 1, 2, 3, 4, 6$. Moreover

$$X = 2Y - Z.$$ 

Problem 2: Two dice are rolled. Let $X$ be equal to the product of the dice minus their sum. Compute $p_i = P(X = i)$ for all values $i = 10, 11, 12$.

Solution: Writing out all possibilities (say in matrix form) and inspecting the possible values for $X$ one finds

$$p_{-1} = \frac{11}{36}; p_0 = p_8 = p_{15} = p_{24} = \frac{1}{36};$$

$$p_1 = p_2 = p_4 = p_5 = p_7 = p_9 = p_{11} = p_{14} = p_{19} = \frac{1}{18}; p_3 = \frac{1}{12}.$$ 

Problem 3: Let $X$ denote the number of heads showing up in 10 independently tossed coins. Compute the probability mass function of $X$.

Solution: $X$ takes the value $k = 0, 1, 2, ..., 10$ and in order to have $k$ heads tossed we need to consider the event $E = \{X = k\}$. But

$$P(E) = \binom{10}{k} 2^{-10}$$

for $k = 0, ..., 10$, hence

$$p(k) = P(X = k) = \binom{10}{k} 2^{-10}.$$
**Problem 4:** Let $X$ be a random variable with probability mass function $p(-2) = \frac{1}{3}$, $p(3) = \frac{1}{4}$, $p(5) = \frac{1}{5}$ and $p(0) = \frac{13}{60}$. Find $E[X]$!

Solution:

\[ E[X] = -\frac{2}{3} + \frac{3}{4} + \frac{5}{5} = \frac{13}{12} \]

**Problem 5:** Five distinct numbers are randomly distributed to five players, numbered 1, 2, 3, 4, and 5. Player 1 compares his number with players 2 to 5 until he has a lower figure than his opponent. Let $X$ denote the number of times player 1 can compare his number until having a lower number. Find $P\{X = i\}$ for $i = 1, ..., 4$ and $E[X]$.

Solution: $i = 1$: The total number of distribution of the five numbers is $5!$. Among these, pair those two where the first 2 numbers are interchanged. Exactly one of them belongs to the event $\{X = 1\}$, so $P(\{X = 1\}) = \frac{1}{2}$.

$i = 2$: Consider all distributions of the five numbers where the first three figures are just permuted. Of these $3! = 6$ possibilities only one belongs to the event $\{X = 2\}$, hence $P(\{X = 2\}) = \frac{1}{6}$.

$i = 3$: In this case consider all distributions of the five numbers where the first 4 figures are permuted. Among these $4!$ distributions there are 2 which belong to the event $\{X = 3\}$, hence $P(\{X = 3\}) = \frac{2}{4!} = \frac{1}{12}$.

$i = 4$: In this case player 1 has to receive the highest number, or he has to receive the second highest number and the 5th player the highest one. It follows that $P(\{X = 4\}) = \frac{1}{5} + \frac{1}{20} = \frac{1}{4}$.

**Problem 6:** A metro runs 5 times on a weekday between 8am and 9am. On average the trains no 1 to 5 contain 600, 800, 700, 400 and 900 passengers, respectively. Let $Y$ be the number of passengers in a randomly chosen train, and let $X$ be the number of passengers in the train of a randomly chosen passenger. Compute $E(X)$ and $E(Y)$. Why do people expect the first expectation to be larger?

Solution:

\[ E[Y] = \frac{1}{5}(600 + 800 + 700 + 400 + 900) = 680. \]

The probability that the train number of a randomly chosen passenger is $k$ (for $k = 1, 2, 3, 4, 5$) is $\frac{3}{17}, \frac{4}{17}, \frac{7}{34}, \frac{2}{17}$ and $\frac{9}{34}$ respectively. Hence

\[ E[X] = 600 \cdot \frac{3}{17} + 800 \cdot \frac{4}{17} + 700 \cdot \frac{7}{34} + 400 \cdot \frac{2}{17} + 900 \cdot \frac{9}{34} = 723.5. \]

**Problem 7:** An insurance company writes a policy that $500 are paid if a bicycle is stolen in a year. It is known that a bicycle is stolen with probability 0.05. What should
the company charge per year in order to make a 10 percent profit on the average?

Solution: Let $X$ be the random variable which is $-500$ if the bicycle is stolen and $0$ if it not stolen. We want to determine the premium $p$ so that the expected profit is 10 percent of the premium. The profit per year is

$$Z = X + p$$

so

$$E[Z] = E[X] + p = -500 \times 0.05 + p = p - 25.$$ 

In order to make 10 percent profit, $p$ is determined by the equation

$$p - 25 = 0.1p$$

or

$$p = $27.78$$

**Problem 8:** A random variable $X$ takes the values $1$, $3$, $5$, $-2$ and $-4$ with probabilities $P(X = 1) = \frac{43}{120}$, $P(X = 3) = \frac{1}{6}$, $P(X = 5) = \frac{1}{17}$, $P(X = -2) = \frac{1}{4}$ and $P(X = -4) = \frac{1}{8}$. Compute the variance of $X$.

Solution: We have

$$E[X] = \frac{43}{120} + \frac{1}{2}(1 + 1 - 1 - 1) = \frac{43}{120}. $$

$$E[X^2] = \frac{43}{120} + \frac{1}{2}(3 + 5 - 2 - 4) = \frac{43}{120} + 7.$$ 

Hence

$$Var[X] = E[X^2] - (E[X])^2 = \frac{43}{120}(1 - \frac{43}{120}) + 1 = 7.23.$$ 

**Problem 9:** A random variable $X$ has two possible values, $a$ and $b$, each of them is attained with probability $\frac{1}{2}$. If it is known that $E[X] = 1$ and $Var[X] = 5$, find all possible values of the pair $(a, b)$.

Solution:

$$1 = E[X] = \frac{a + b}{2}$$

and

$$5 = Var[X] = \frac{1}{2}((a - \frac{a + b}{2})^2 + (b - \frac{a + b}{2})^2).$$

Solve these equations for $a$ and $b$ to find

$$a = 1 + \sqrt{5} \quad b = 1 - \sqrt{5}$$
Problem 10: A sample of 10 is taken from a production line of 100 items. Assume that it is known that 1 percent of the production is defective. Find the expected number of defective items in the sample of 10.

Solution: Let $X_l$ be the random variable which is one if the $l$-th item is defective and zero otherwise. Then the number of defective items in the sample of 10 is

$$X = X_1 + X_2 + ... + X_{10},$$

and

$$E[X] = E[X_1] + E[X_2] + ... + E[X_{10}].$$

since each $X_l$ takes only two values, with probability 0.01 to be equal to one,

$$E[X_l] = 0.01$$

for each $l = 1, ..., 10$. Therefore

$$E[X] = 10 \times 0.01 = 0.1.$$