Solution

**Problem 1:** Use the comparison test to show that the following series converge:

(a) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \]

(b) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n} \]

You may use \( \ln x < x \) for \( x > 0 \).

**Solution**

For both series we show that 14.6 (i), the comparison test, can be applied.

(a) Let \( b_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \geq 0 \). Define \( a_n = n^{-3/2} \). Then

\[ |b_n| = b_n = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq a_n, \]

and we need to observe that \( \sum \frac{1}{n^{3/2}} \) is convergent.

In order to see that \( \sum a_n \) converges, use comparison test:

For \( m^2 \leq n < (m + 1)^2 \) we have \( a_n \leq \frac{1}{m^3} \) and there are \( 2m + 1 \) indices \( n \) between \( m^2 \) and \( (m + 1)^2 \). Hence

\[ \sum a_n \leq \sum \frac{2m + 1}{m^3} \leq 3 \sum \frac{1}{m^2}. \]

(b) Let \( b_n = \frac{1}{n^2 - \ln n} \). Define \( a_n = \frac{2}{n^2} \). Then

\[ |b_n| = b_n \leq a_n, \]

and \( 2 \sum \frac{1}{n^2} \) is a convergent series.

**Problem 2:** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions. Prove, using the sequence definition, that if \( f, g \) are continuous at \( x \), then

1. \( h := f + g \) is continuous at \( x \).
2. \( h := f - g \) is continuous at \( x \).
3. \( h := fg \) is continuous at \( x \).

**Solution**
1. Let \( x_n \) be a sequence converging to \( x \). Since \( f \) and \( g \) are continuous at \( x \) we get
\[
(f+g)(x) = f(x)+g(x) = \lim f(x_n) + \lim g(x_n) = \lim f(x_n) + g(x_n) = \lim (f+g)(x_n).
\]

2. Let \( x_n \) be a sequence converging to \( x \). Since \( f \) and \( g \) are continuous at \( x \) we get
\[
(f-g)(x) = f(x)-g(x) = \lim f(x_n) - \lim g(x_n) = \lim f(x_n) - g(x_n) = \lim (f-g)(x_n).
\]

3. Let \( x_n \) be a sequence converging to \( x \). Since \( f \) and \( g \) are continuous at \( x \) we get
\[
(fg)(x) = f(x)g(x) = \lim f(x_n) \lim g(x_n) = \lim f(x_n)g(x_n) = \lim (fg)(x_n).
\]

**Problem 3:** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Prove, using the \( \epsilon - \delta \) definition of continuity that if \( f(x) \neq 0 \) then there is some interval of the form \((x-\delta, x+\delta)\) where \( f \) is non-zero.

**Solution:**

Let \( |f(x)| = a > 0 \). Then there exists \( \delta > 0 \) such that for \( |x - y| < \delta \) we have
\[
|f(y) - f(x)| < \frac{a}{2}.
\]

Therefore, if \( y \in (x-\delta, x+\delta) \), \( |f(y)| \geq |f(x)| - |f(x) - f(y)| \geq \frac{a}{2} > 0 \).

**Problem 4:** Let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions. Use the previous exercise to show that if \( f, g \) are continuous at \( x \), and \( g(x) \neq 0 \) then \( h(x) := f(x)/g(x) \) is well-defined in some interval of the form \((x-\delta, x+\delta)\) and is continuous at \( x \).

**Solution:**

By the previous exercise there is \( \delta > 0 \) such that \( g \) is nonzero on the interval \((x-\delta, x+\delta)\). Therefore the function \( f/g \) is well defined on that interval. In order to show continuity at \( x \), let \( x_n \) be a sequence in that interval converging to \( x \). Then we can apply Theorem 9.6 to obtain
\[
\lim \frac{f}{g}(x_n) = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(x)}{g(x)} = \frac{f}{g}(x).
\]

**Problem 5:** Prove that \( f(x) = \cos x \) is continuous at every \( x \in \mathbb{R} \).

**Solution:**

Let \( x_n \) be a sequence converging to \( x \). Let \( \eta_n = x_n - x \). Then
\[
\cos(x + \eta_n) = \cos x \cos \eta_n - \sin x \sin \eta_n.
\]

We know that \( \sin \) is a continuous function at \( x = 0 \), hence \( \lim \sin \eta_n = 0 \). We also know that \( \cos \eta_n = \pm \sqrt{1 - \sin^2 \eta_n} \) is continuous at \( 0 \), hence \( \lim \cos \eta_n = 0 \) and
\[
\lim \cos x_n = \lim \cos(x + \eta_n) = \cos x.
\]

**Problem 6:** Let \( f, g \) be two continuous functions on \([a, b] \), and assume that \( f(a) < g(a) \), but \( f(b) > g(b) \). Prove that \( f(x) = g(x) \) for some \( a < x < b \).
Solution:

Take \( h(x) = f(x) - g(x) \) for \( a \leq x \leq b \). Then \( h(a), 0 \) and \( h(b) > 0 \), hence by the intermediate value theorem there is a point \( x \in (a, b) \) with \( h(x) = 0 \). This means \( f(x) = g(x) \).

**Problem 7:** Prove that every function \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^n \) with \( n \in \mathbb{N} \), is continuous at every \( x \in \mathbb{R} \).

Solution:

The proof is by induction on \( n \in \mathbb{N} \) for the statements

\( P_n: \) the function \( f_n(x) = x^{n-1} \) is continuous at every \( x \in \mathbb{R} \).

We know that \( f_1 \) and \( f_2 \) are continuous at every point in the real line. This implies that \( P_1 \) is true.

Induction step: Suppose that \( f_n \) is continuous at every point in the real line. Since \( f_{n+1} = f_n f_1 \) and since \( f_1 \) is as well continuous, we obtain for a sequence \( x_m \) converging to an arbitrary \( x \in \mathbb{R} \) that

\[
f_{n+1}(x) = f_n(x)f_1(x) = \lim_{m \to \infty} f_n(x_m) \lim_{m \to \infty} f_1(x_m) = \lim_{m \to \infty} f_n(x_m) f_1(x_m) = \lim_{m \to \infty} f_{n+1}(x_m).
\]