Exercise 1.1: We show the statement by mathematical induction. The statements are

- $P_1: 1^2 = \frac{1}{6}1(1 + 1)(2 \cdot 1 + 1)$
- $P_2: 1^2 + 2^2 = \frac{1}{6}2(2 + 1)(2 \cdot 2 + 1)$
- ...
- $P_n: 1^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2 \cdot n + 1)$
- ...

It is sufficient to show that $P_1 \Rightarrow P_2 \Rightarrow \ldots$ and that $P_1$ is true. Then, $\forall n \in \mathbb{N}$, $P_n$ is true.

Basis $P_1$: Since $1^2 = 1$ and $\frac{1}{6}1(1 + 1)(2 \cdot 1 + 1) = 1$ are true statements, also the statement $1^2 = \frac{1}{6}1(1 + 1)(2 \cdot 1 + 1)$ is a true statement because "=" is an equivalence relation. Therefore $P_1$ is true.

Induction hypothesis: $\forall k \leq n$ the statement $P_k$ is true.

Induction step: We begin with the true statement $P_n$:

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2 \cdot n + 1).$$

Adding $(n + 1)^2$ on each side yields the true statement $P_{n+1}$

$$1^2 + 2^2 + \ldots + n^2 + (n + 1)^2 = \frac{1}{6}n(n + 1)(2 \cdot n + 1) + (n + 1)^2.$$

We transform the right hand side of equation into an equivalent expression:

$$\frac{1}{6}n(n + 1)(2 \cdot n + 1) + (n + 1)^2$$

$$= (n + 1) \left[ \frac{1}{6}n(2 \cdot n + 1) + (n + 1) \right]$$

$$= (n + 1) \left[ \frac{1}{6}(2n^2 + n + 6n + 6) \right]$$

$$= (n + 1) \left[ \frac{1}{6}(n + 2)(2n + 3) \right]$$

$$= \frac{1}{6}(n + 1)(n + 2)(2 \cdot (n + 1) + 1).$$
This means that we get the true statement

\[ 1^2 + 2^2 + ... + n^2 + (n + 1)^2 = \frac{1}{6}(n + 1)(n + 2)(2 \cdot (n + 1) + 1), \]

which is \( P_{n+1} \) and the induction step has been proven.

**Exercise 1.3:** We show the statement by mathematical induction. The statements are

- \( P_1: 1^3 = (1)^2 \)
- \( P_2: 1^3 + 2^3 = (1 + 2)^2 \)
- ...
- \( P_n: 1^3 + ... n^3 = (1 + 2 + ... + n)^2 \)
- ...

It is sufficient to show that \( P_1 \Rightarrow P_2 \Rightarrow ... \) and that \( P_1 \) is true. Then, \( \forall n \in \mathbb{N}, P_n \) is true.

**Basis \( P_1 \):** This is obvious since \( 1^2 = 1^3 \).

**Induction hypothesis:** \( \forall k \leq n \) the statement \( P_k \) is true.

**Induction step:** We transform the right hand side of \( P_{n+1} \) into an equivalent expression, using that \( 1^3 + ... n^3 = (1 + 2 + ... + n)^2 \) and the binomial formula:

\[
(1 + 2 + ... + n + (n + 1))^2 = (1 + 2 + ... + n)^2 + 2(1 + ... + n)(n + 1) + (n + 1)^2 \\
= 1^3 + 2^3 + ... + n^3 + (n + 1)\left[ 2(1 + 2 + ... + n) + (n + 1) \right].
\]

Next we transform further using the true statement from Exercise 1.1:

\[ 1 + 2 + ... + n = \frac{n(n + 1)}{2}. \]

It follows that

\[
(1 + 2 + ... + n + (n + 1))^2 = 1^3 + 2^3 + ... + n^3 + (n + 1)\left[ n(n + 1) + (n + 1) \right] \\
= 1^3 + 2^3 + ... + n^3 + (n + 1)^3.
\]

This is \( P_{n+1} \) and the induction step has been proven.

**Exercise 1.11:** (a) Let \( P_n \) be true. We need to show that

\[ (n + 1)^2 + 5(n + 1) + 1 \]
is an even integer. Observe that
\[(n + 1)^2 + 5(n + 1) + 1 = n^2 + 2n + 1 + 5n + 5 + 1 = n^2 + 5n + 1 + 6.\]

Since by \(P_n\) we assume that \(n^2 + 5n + 1\) is an even number, we have a representation \(n^2 + 5n + 1 = 2p\) for some \(p \in \mathbb{N}\). Therefore
\[(n + 1)^2 + 5(n + 1) + 1 = 2p + 6 = 2(p + 3)\]
and \((n + 1)^2 + 5(n + 1) + 1\) is an even number. So \(P_{n+1}\) holds.

(b) The statement \(P_n\) is not true for \(n = 1\) and all even \(n\). Therefore it is not true for all \(n \in \mathbb{N}\), because if it would be true for the odd \(n\), it must be true for \(n + 1\) which is even, a contradiction. Therefore, it is important to prove the case \(P_1\).

**Exercise 1.12:** (a) For \(n = 1\)
\[a + b = \binom{1}{0}a^0b + \binom{1}{1}a^1b^0.\]
For \(n = 2\)
\[(a + b)^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^0b^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^2b^0,
and observe that \(\binom{2}{1} = 2\). For \(n = 3\)
\[(a + b)^3 = (a + b)(a + b) = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3;
and
\[\binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3, \binom{3}{3} = 1\]
inserted in the right hand side of \(P_3\) yields the same expression.

(b) Let \(n \in \mathbb{N}\) and \(1 \leq k \leq n, k \in \mathbb{N}\). Then
\[
\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} = \frac{n!}{(k-1)!(n-k)!} \left( \frac{1}{k} + \frac{1}{n-k+1} \right) = \frac{n!}{(k-1)!(n-k)!} \frac{n-k+1+k}{k(n-k+1)} = \frac{n!(n+1)}{(k-1)!(n-k)!k(n+1-k)} = \binom{n+1}{k}.
\]
This chain consists of 5 equations, all of them are true, so the first expression equating the last expression is as well true.

(c) We verify the binomial formula by induction. Let \( P_n \) be the statement

\[
(a + b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}ab^{n-1} + ... + \binom{n}{n}a^nb^0.
\]

Basis \( P_1 \): holds because

\[
a + b = \binom{1}{0}a^0b + \binom{1}{1}ab^0
\]

is true (see (a)).

Induction hypothesis: Suppose \( P_k \) holds for \( k = 1, \ldots, n \).

\[
P_k : (a + b)^k = \binom{k}{0}a^0b^k + \binom{k}{1}ab^{k-1} + ... + \binom{k}{k}a^kb^0.
\]

Induction step:

\[
(a + b)^{n+1} = (a + b)^n(a + b)
\]

\[
= \left( \binom{n}{0}a^0b^n + \binom{n}{1}ab^{n-1} + ... + \binom{n}{n}a^nb^0 \right) (a + b)
\]

\[
= \left( \binom{n}{0}a^1b^{n-1} + \binom{n}{1}a2b^{n-2} + ... + \binom{n}{n}a^{n+1}b^0 \right)
\]

\[
+ \left( \binom{n}{0}a^0b^{n+1} + \binom{n}{1}ab^n + ... + \binom{n}{n}a^nb^1 \right)
\]

\[
= \binom{n+1}{0}a^0b^{n+1} + \left( \binom{n+1}{1} + \binom{n}{0} \right) a^1b^n + \left( \binom{n+1}{2} + \binom{n}{1} \right) a^2b^{n-1} + ... + \binom{n}{n}a^{n+1}b^0
\]

where we used (b) for the last equation. Since \( \binom{n+1}{0} = \binom{n+1}{n} = \binom{n}{n} = \binom{n}{0} = 1 \), we obtain

\[
\binom{n}{0}a^0b^{n+1} + \left( \binom{n+1}{1} + \binom{n}{0} \right) a^1b^n + \left( \binom{n+1}{2} + \binom{n}{1} \right) a^2b^{n-1} + ... + \binom{n}{n}a^{n+1}b^0
\]

\[
= \binom{n+1}{0}a^0b^{n+1} + \binom{n+1}{1}a^1b^n + \binom{n+1}{2}a^2b^{n-1} + ... + \binom{n+1}{n}a^{n+1}b^0.
\]

This concludes the proof.

**Exercise 2.3:** We first plan to find an equation

\[
a_nx^n + a_{n-1}x^{n-1} + ... + a_0x^0 = 0
\]
for which \((2 + \sqrt{2})^{1/2}\) is a solution (thus it is an algebraic number). First note that \((2 + \sqrt{2})^{1/2}\) is a solution of the equation \(x^2 - 2 - \sqrt{2} = 0\). Hence \((2 + \sqrt{2})^{1/2}\) is also a solution of the equation
\[(x^2 - 2)^2 - 2 = 0.\]
The left hand side can be rewritten in the form \(x^4 - 2x^2 - 2\), hence \((2 + \sqrt{2})^{1/2}\) is a solution of the equation
\[x^4 - 2x^2 - 2 = 0.\]

We show next that \((2 + \sqrt{2})^{1/2}\) is not rational. We do this by indirect proof. Assume therefore that “\((2 + \sqrt{2})^{1/2}\) is rational” is a true statement. We also know that “\((2 + \sqrt{2})^{1/2}\) is a solution of the equation \(x^4 - 2x^2 - 2 = 0\)” is also a true statement. We need to show that this implies that “\((2 + \sqrt{2})^{1/2}\) is not a solution” is a correct statement, which is a contradiction, since a statement and its negation cannot be true at the same time (by assumption of our elementary logic). It follows that the assumption made cannot be correct (since all other assumptions are known to be correct). So we get that \(\neg(2 + \sqrt{2})^{1/2}\) is rational”, i.e. \(\neg(2 + \sqrt{2})^{1/2}\) is not rational”, is a correct statement.

In order to do this, let \(\frac{p}{q} = (2 + \sqrt{2})^{1/2}\) be rational, where \(p\) and \(q\) have no common divisor. Using Theorem 2.2 of Ross we then get that “\(q\) divides \(a_n = 1\)” and “\(p\) divides \(a_0 = -2\)” are correct statements. It follows that \(q = \pm 1\) and \(p = \pm 2\) or \(p = \pm 1\), leading to the fact that \(\pm 2\) or \(\pm 1\) are possible solutions of the equation. However,

\[(\pm 2)^4 - 2(\pm 2)^2 - 2 = 16 - 8 - 2 = 6 \neq 0\]

and

\[(\pm 1)^4 - 2(\pm 1)^2 - 2 = 1 - 2 - 2 = -3 \neq 0\]
telling that neither of the possibilities \(\pm 2\) and \(\pm 1\) is a solution of the equation.

This finished the proof.

**Exercise 3.3:**

(iv) By M2 (p. 13 in Ross) we get

\[a(-b) = (-b)a.\]

By (iii) of Theorem 3.1 (p. 15 in Ross) we have

\[(-b)a = -ba.\]

Setting \(b = 1\) and using M2,M3 the last equation reads

\[-a = (-1)a = a(-1).\]

Therefore, by M1,M2

\[(-a)(-b) = (a(-1))(b(-1)) = ab(-1)(-1).\]
Applying this equation for $a = b = 1$ we get, using $M1,M2,M3$

$$(-1)(-1) = (1)(1)(-1)(-1) = ((1)(-1))((1)(-1)) = (1)(1) = 1.$$ 

Thus

$$ab(-1)(-1) = ab1 = ab,$$

where we used again $M1,M3$. The claim follows.

(v) Since $ac = bc$, we get by $M4$ that

$$(ac)(c^{-1}) = (bc)(c^{-1}).$$

(note that $ac$ is the same element as $bc$, so we can replace it because multiplication is defined on the set of real numbers). By $M1,M3,M4$

$$a = a(1) = a(cc^{-1}) = (ac)(c^{-1}) = (bc)(c^{-1}) = b(cc^{-1}) = b(1) = b.$$

This proves the claim.

Exercise 3.4:

(v) Since by Theorem 3.2 (iv) (Ross p. 15) with $a = 1$ and $M3$

$$0 \leq 1^2 = 1,$$

it suffices to show that $0 \neq 1$. Now, if $0 = 1$ then by $M3$ and Theorem 3.1 (ii) for any $a \neq 0$ (which exists)

$$a = 1 \cdot a = 0 \cdot a = 0,$$

a contradiction.

(vii) Let $0 < b$. By (vi), $0 < b^{-1}$. Now let $a < b$. Then by $O5$

$$aa^{-1} \leq ba^{-1}$$

so by $M2, M4$

$$1 \leq a^{-1}b$$

and by $O5, M3, M4$ again

$$b^{-1} \leq a^{-1}.$$