Problem 1: Determine which of the following series converge. Prove your claim.

1. $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 7}$
2. $\sum_{n=1}^{\infty} \frac{3n^3}{n^5}$
3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3 + 7}$

Problem 2: Prove for two convergent series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ that

$$\sum_{n=m}^{\infty} (a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$  

Problem 3: Let $\sum_{n=m}^{\infty} a_n$ be absolutely convergent and let $(b_n)_{n=m}^{\infty}$ be a bounded sequence. Show that the sequence $\sum_{n=m}^{\infty} a_nb_n$ converges.

Problem 4: Show that for each sequence $(a_n)_{n=1}^{\infty}$, where $a_n$ is either 0 or 1, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

converges to some real number in the unit interval $[0, 1]$ (dual expansion of reals).

Problem 5: Prove that for any sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ that satisfies

$$|a_n - a_{n+1}| \leq \frac{1}{2^n} \quad \forall n,$$

the series $\sum a_n$ is convergent.

Problem 6: Let $a_{n+1} = 1 - \sqrt{1 - a_n}$ for $n \geq 1$ and $0 \leq a_1 < 1$. Show that the series $\sum a_n$ converges.

Problem 7: Let $0 < b_1 < a_1$ and set

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_nb_n}.$$  

(a) Show that $(b_n)$ is nondecreasing and bounded above and that $(a_n)$ is nonincreasing and bounded below.

(b) Prove that $\sum (a_n - b_n)$ is convergent.