Due Date: Monday, 01/25/2010 in class.

Solve as many problems as you can. Only one problem will be graded in detail. Full credit for 3 solved problems. Any of the remaining problems may appear on a midterm or on the final exam. You are required to supply complete proofs, similar in nature and form to those given in class or in your textbook.

Problem 1: Do Exercise 1.1 in the textbook:

Prove that \(1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)\) for all natural numbers \(n\).

Problem 2: Do Exercise 1.3 in the textbook:

Prove \(1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2\) for all natural numbers \(n\).

Problem 3: Do Exercise 1.11 in the textbook:

For each \(n \in \mathbb{N}\), let \(P_n\) denote the assertion “\(n^2 + 5n + 1\) is an even integer”.

(a) Prove that \(P_{n+1}\) is true whenever \(P_n\) is true.

(b) For which \(n \in \mathbb{N}\) is \(P_n\) actually true? What is the moral of this exercise?

Problem 4: For \(n \in \mathbb{N}\), let \(n!\) denote the product \(1 \cdot 2 \cdot 3 \cdots \cdot n\). Also let \(0! = 1\) and define

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } k = 0, 1, \ldots, n.
\]

The binomial theorem asserts that

\[
(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.
\]

(a) Verify the binomial theorem for \(n = 1, 2,\) and 3.

(b) Show that \(\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}\) for \(k = 1, 2, \ldots, n\).

(c) Prove the binomial theorem using mathematical induction and parts (a) and (b).

Problem 5: Do Exercise 2.3 in the textbook:

Show that \((2 + \sqrt{2})^{1/2}\) does not represent a rational number.

Problem 6: Do Exercise 3.3 in the textbook:

Prove (iv) and (v) of Theorem 3.1 in class (and in the textbook).

Problem 7: Do Exercise 3.4 in the textbook:

Prove (v) and (vii) of Theorem 3.2 in class (and in the textbook).