Aspect ratio dependence of electromechanical coupling coefficient of piezoelectric resonators

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The most important parameter characterizing a piezoelectric material is the electromechanical coupling coefficient, which specifies the conversion efficiency between electrical and mechanical energies for a given vibration mode. For the resonance along the poling direction, there are two coupling coefficients defined, i.e., $k_{33}$ and $k_{31}$, which are for resonators of the same mode but two aspect ratios and they differ substantially. We have derived a unified formula for this coupling coefficient as a function of the vibrator aspect ratio. The unified formula can provide more accurate description to the effective coupling coefficient of resonators not satisfying the extreme aspect ratio requirements. © 2005 American Institute of Physics [DOI: 10.1063/1.2053366]

The electromechanical coupling coefficient of piezoelectric materials is one of the most important parameters for modeling and designing of piezoelectric devices, such as ultrasonic transducers, pressure sensors, and piezoelectric actuators. The most commonly used mode of vibration in piezoelectric devices is the resonance along the poling direction. It is well known that the electromechanical coupling coefficient depends strongly on the aspect ratio of the resonator. For the two extreme cases, i.e., with the aspect ratios $0$ and $\infty$, the equation of motion can be approximated by a one-dimensional (1D) equation.\(^1\) If we define the aspect ratio as the dimension of poling direction over the lateral dimension, then the aspect ratio $\rightarrow \infty$ represents a long bar poled along the long axis, which yields an electromechanical coupling coefficient of $k_{33}$. The $0$ aspect ratio case, on the other hand, describes a thin plate, which has an electromechanical coupling coefficient $k_1$. The two coupling coefficients $k_{33}$ and $k_1$ differ substantially, for example, $k_{33} = 0.705$ but $k_1 = 0.486$ for the well-known piezoelectric ceramic Pb(Zr,Ti)O$_3$ [PZT]. It is easy to see that the only difference between the two cases is the aspect ratio since the vibration mode is exactly the same.

When a vibrator does not have the specified extreme aspect ratios, neither $k_{33}$ nor $k_1$ can describe the energy conversion efficiency. This means that modeling using either $k_{33}$ or $k_1$ will introduce large errors compared to the performance of real devices. For this purpose, it is practically useful to derive a general formula that can describe the coupling efficiency of vibrators with any given aspect ratio.

Based on mode coupling analysis, we have derived here a unified formula for the electromechanical coupling coefficient that can be used for longitudinal vibrators of any aspect ratio. For simplicity, we analyze a cylindrical resonator as shown in Fig. 1 so that the aspect ratio $G$ can be defined as $G = H/R$, where $H$ is the height along the poling direction and $R$ is the radius of the cylinder. For convenience, we still use Cartesian coordinates to write down the constitutive relations.

The electrical boundary conditions for this vibrator are:

$$D_z = D_y = E_z = E_y = 0.$$  

Because there are no shear vibrations involved, we can write the constitutive relations as follows:

$$
S_1 = s_{11}^E T_1 + s_{12}^E T_2 + s_{13}^E T_3 + d_{31} E_z, \\
S_2 = s_{12}^E T_1 + s_{11}^E T_2 + s_{13}^E T_3 + d_{31} E_z, \\
S_3 = s_{13}^E T_1 + s_{11}^E T_2 + s_{13}^E T_3 + d_{33} E_z, \\
D_z = d_{33}(T_1 + T_2) + d_{33} T_3 + E_z E_z.
$$

The internal energy of a linear system is given by\(^1\)

$$U = \frac{1}{2} S_i T_i + \frac{1}{2} D_m E_m, \quad i = (1 - 6), \quad m = (x,y,z).$$

Substituting Eqs. (1a)–(1d) into Eq. (2) we have the following form for the internal energy:

$$U = \frac{1}{2} [S_1 T_1 + S_2 T_2 + S_3 T_3] + \frac{1}{2} [D_z E_z] = U_x + 2U_m + U_d.$$  

The electromechanical coupling coefficient is defined as

![FIG. 1. Schematic plot of the cylindrical piezoelectric resonator under investigation.](image-url)
where $U_r$, $U_d$, and $U_m$ are the elastic, dielectric, and coupling energies, respectively. With the conditions $S_1=S_2$, $T_1=T_2$, Eqs. (1a), (1c), and (1d) can be rewritten as

$$S_1 = (s_{11}^{E} + s_{12}^{E})T_1 + s_{13}^{E}T_3 + d_{13}E_z,$$

$$S_3 = 2s_{13}^{E}T_1 + s_{33}^{E}T_3 + d_{33}E_z,$$

$$D_z = 2d_{12}T_1 + d_{33}T_3 + E_z.$$

Eliminating $T_1$ in Eqs. (5b) and (5c) using Eq. (5a), $S_3$ and $D_z$ become

$$S_3 = \frac{2s_{13}^{E}}{(s_{11}^{E} + s_{12}^{E})}(S_1 - s_{13}^{E}T_3 - d_{13}E_z) + s_{33}^{E}T_3 + d_{33}E_z,$$

$$D_z = \frac{2d_{12}}{(s_{11}^{E} + s_{12}^{E})}(S_1 - s_{13}^{E}T_3 - d_{13}E_z) + d_{33}T_3 + E_z.$$

Hence, the internal energy is now a function of $S_1$, $T_3$, and $E_z$.

$$U = \frac{1}{2}\left[\frac{2}{(s_{11}^{E} + s_{12}^{E})}S_1^2 + \left(\frac{s_{13}^{E} - 2(s_{11}^{E})^2}{(s_{11}^{E} + s_{12}^{E})}\right)T_3^2\right] + \frac{1}{2}\left[2d_{33} - \frac{2s_{33}^{E}d_{13}}{(s_{11}^{E} + s_{12}^{E})}\right]T_3E_z + \left(s_{zz}^{E} - \frac{2d_{12}^{E}}{(s_{11}^{E} + s_{12}^{E})}\right)E_z^2.$$

For very large aspect ratio resonators, $T_1 = T_2 \sim 0$, so that Eq. (1a) becomes

$$S_1^* = s_{13}^{E}T_3 + d_{13}E_z.$$

For small aspect ratio resonators, we do not have such a simple case since the stress $T_1$ is finite, hence, we must approach the problem from a different angle. When the fundamental longitudinal and radial modes are well separated, their resonance frequencies (note, this frequency corresponds to the antiresonant frequency for the thickness mode but the resonant frequency for the radial mode) are given by

$$f_H = \frac{1}{2H} \sqrt{\frac{c_{33}^{D}}{\rho}} \frac{\nu_3}{2 \pi},$$

and

$$f_R = \frac{\eta_1}{2R\pi} \sqrt{\frac{1}{\rho c_{11}^{E}(1-\sigma)}} = \frac{\eta_1 \nu_1}{2\pi R \sqrt{(1-\sigma)}},$$

where $\nu_1$ and $\nu_3$ are phase velocities along $x_1$ and $x_3$ directions, $\sigma = -s_{12}^{E}/s_{11}^{E}$ is the Poisson’s ratio and $\eta_1 (=2$ for most practical cases) is the first root of the Bessel’s equation

$$\eta J_0(\eta) = (1-\sigma)J_1(\eta).$$

For the small aspect ratio case, i.e., thin disk, the ratio of the displacements caused by the two resonances on average is proportional to the frequency ratio,

$$\frac{\xi_1}{\xi_3} \approx \frac{\nu_1}{\nu_3} \frac{f_R}{f_H} = \frac{\pi R \sqrt{(1-\sigma)} f_R}{\eta_1 H f_H} \frac{f_R}{f_H}.$$

Therefore,

$$S_3 \approx \frac{f_R}{f_H} S_1.$$

For the plate mode, the strain $S_1$ is nearly a constant while the strain $S_1$ will strongly depend on the ratio of two resonance frequencies and its value changes from $-1$ to $0$. $c_{11}^{E} + c_{12}^{E}$, as the aspect ratio $G$ increases from near $0$ to $\sim \infty$ (very large). Based on Eqs. (9) and (12), we propose the following functional form for the strain $S_1$:}

$$S_1 = (s_{13}^{E}T_3 + d_{13}E_z)\kappa(G)\frac{f_1}{f_2}.$$

Here $\kappa(G)$ is a function of the aspect ratio $G$ to be determined.

We use $f_1/f_2$ instead of $f_R/f_H$ in the proposed form to include the mode coupling effect. For coupled vibrations, the resonant frequencies of the two modes are given by the solutions of following equation:

$$(f_H)^2 - (f_H)^2(1-\sigma) = \Gamma^2 (f_H)^2(f_R)^2,$$

where $f_H$ and $f_R$ are, respectively, the resonant frequencies along the radial and poling directions without mode coupling. The thickness mode resonance $f_H$, radial mode resonance $f_R$, and mode-coupling constant $\Gamma$ are given by the following equations:

$$f_H = \frac{1}{H} \sqrt{\frac{c_{33}^{D}}{\rho}} \frac{\nu_3}{2 \pi},$$

and

$$f_R = \frac{1}{2R \pi} \sqrt{\frac{c_{11}^{E}}{\rho}}.$$
Note that the frequency \( f_1 \) is always lower than \( f_2 \) as shown in Fig. 2, and hence, always represents the resonant frequency of the larger dimension. In other words, for the case of PZT, if the aspect ratio is smaller than 1.1599, \( f_1 \) represents the vibration frequency of the radial mode, while for aspect ratio larger than 1.1599, it represents the vibration frequency of the longitudinal mode along \( x_3 \).

In order to determine the coefficient form \( \kappa(G) \) in Eq. (14) we take the limit of \( G \to \infty \) so that the strain \( S_1 \) will recover \( S_1^\ast \) of Eq. (9):

\[
\lim_{G \to \infty} \frac{f_1}{f_2} = \lim_{G \to \infty} \frac{2X_t \kappa(G)}{G \xi} \sqrt{\frac{c_d^{\epsilon}}{c_{11} (1 - \Gamma^2)}} = 1.
\]

This requires that \( \kappa(G) \) must have the following form:

\[
\kappa(G) = \frac{G \xi}{2X_t} \sqrt{\frac{c_d^{\epsilon}}{c_{11} D (1 - \Gamma^2)}}.
\]

Therefore, the aspect-ratio-dependent lateral strain \( S_1 \) is given by

\[
S_1 = \frac{G \xi}{2X_t} \sqrt{\frac{c_d^{\epsilon}}{c_{11} D (1 - \Gamma^2)}} \frac{f_1}{f_2} (\gamma_{13} T_3 + d_{13} E_c).
\]

\[
g(G) = \frac{G \xi}{2X_t} \sqrt{\frac{c_d^{\epsilon}}{c_{11} D (1 - \Gamma^2)}} \frac{f_1}{f_2}.
\]

Now we can substitute Eq. (24) into Eq. (8) and then into Eq. (4) which leads to the final form of the unified formula for the electromechanical coupling constant:

and the factor \( \zeta \) is the first root of Bessel’s function that satisfies the following equation:

\[
\zeta J_0(\zeta) = \left( 1 - \frac{c_{12}^E}{c_{11}^E} \right) J_1(\zeta).
\]

Solving Eq. (15) leads to the following solutions:

\[
f_1 H = \sqrt{\frac{1}{8 \pi^2 \rho} \left[ 4c_{33} D x_t^2 + c_{11}^E c_d^{\epsilon} G^2 - \sqrt{16c_{33}^2 x_t^4 - 8c_{11}^E c_{33} D x_t^2 c_d^{\epsilon} (1 - 2\Gamma^2) G^2 + (c_{11}^E)^2 \xi^4 G^4} \right]},
\]

\[
f_2 H = \sqrt{\frac{1}{8 \pi^2 \rho} \left[ 4c_{33} D x_t^2 + c_{11}^E c_d^{\epsilon} G^2 + \sqrt{16c_{33}^2 x_t^4 - 8c_{11}^E c_{33} D x_t^2 c_d^{\epsilon} (1 - 2\Gamma^2) G^2 + (c_{11}^E)^2 \xi^4 G^4} \right]},
\]

\[
k = \frac{d_{13} + 2d_{13} r_{13}^3 [g^2(G) - 1]}{\sqrt{\left( \frac{c_d^{\epsilon}}{c_{11}^E D} \right)} \left[ g^2(G) - 1 \right]} \left( g_{zz} - \frac{r_{13}^3}{r_{13}^3 + r_{13}^3} \right) = \frac{\xi}{c_{33}^D} = k_t.
\]

When \( G \to 0 \), \( g(G) \to 0 \), Eq. (26) recovers the expression for \( k_t \):

\[
\lim_{G \to \infty} k = \frac{d_{13} - 2d_{13} r_{13}^3 [g^2(G) - 1]}{\sqrt{\left( \frac{c_d^{\epsilon}}{c_{11}^E D} \right)} \left[ g^2(G) - 1 \right]} = \frac{\xi}{c_{33}^D} = k_{zz}.
\]

Using the derived unified formula Eq. (26), we have calculated the aspect ratio dependence of the electromechanical coupling coefficient \( k \) for PZT-5 and BaTiO3 ceramic using the material parameters given in Ref. 1. As shown in Fig. 3, the \( k \) curve is a kink-type solution and changes quickly in the vicinity of \( G = 1.1599 \). For an aspect ratio \( G = 3 \), the \( k \) value is already reaching 98.4% of the \( k_{zz} \) value while for the case of aspect ration \( G = 0.2 \), the \( k \) value is already within 99.9% of the \( k_t \) value.

The unified formula for the electromechanical coupling coefficient can provide an accurate description of the energy conversion efficiency for resonators that do not satisfy the aspect ratio requirements for \( k_{33} \) or \( k_t \) modes. Using this unified formula, there is no need to define those two electromechanical coupling coefficients, \( k_{33} \) and \( k_t \) for the same mode.

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