A Nonlinear Variational Wave Equation

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A nonlinear wave equation

$$u_{tt} - c(u)(c(u)u_x)_x = 0$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)$$

$c : \mathbb{R} \mapsto \mathbb{R}^+$ is a smooth, bounded, uniformly positive function

$$\pm c(u) = \text{wave speeds}$$
Auxiliary variables

\[
\begin{align*}
R & \doteq u_t + c(u)u_x, \\
S & \doteq u_t - c(u)u_x, \\
\end{align*}
\]

Evolution equation for $R, S$:

\[
\begin{align*}
R_t - cR_x &= \frac{c'}{4c} (R^2 - S^2) \\
S_t + cS_x &= \frac{c'}{4c} (S^2 - R^2)
\end{align*}
\]

Possible blow-up: $|R|, |S| \to \infty$ in finite time

\[
c' \equiv 0 \implies \text{D’Alembert solution of wave equation}
\]
Conserved quantities

\[\begin{align*}
R_t - cR_x &= \frac{c'}{4c}(R^2 - S^2) \\
S_t + cS_x &= \frac{c'}{4c}(S^2 - R^2)
\end{align*}\]

Multiply the first equation by \(R\) and the second one by \(S\)

\[\Rightarrow \text{ balance laws for } R^2, S^2:\]

\[\begin{align*}
(R^2)_t - (cR^2)_x &= \frac{c'}{2c}(R^2 S - RS^2) \\
(S^2)_t + (cS^2)_x &= -\frac{c'}{2c}(R^2 S - RS^2)
\end{align*}\]

Conserved quantities:

\[E = \frac{1}{2}(u_t^2 + c^2 u_x^2) = \frac{R^2 + S^2}{4} \quad M = -u_t u_x = \frac{S^2 - R^2}{4c}\]

\(R^2\) and \(S^2\) represent the energy of backward and forward moving waves.

Note: these energies are not separately conserved

Energy is transferred from forward to backward waves, and vice versa.
\[ \int E \, dx = \frac{1}{2} \int (u_t^2 + c^2 u_x^2) \, dx = \text{constant} \]

(for smooth solutions)

\[ 0 < \kappa^{-1} \leq c(u) \leq \kappa \]

Natural domain: \((u, u_x) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\)

\[ \implies \text{solutions remain Hölder continuous} \]
Characteristic Variables

Equations for characteristics

\[ \dot{x}^+ = c(u), \quad \dot{x}^- = -c(u) \]

\[ s \mapsto x^+(s, t, x) \quad x \mapsto x^-(s, t, x) \]

As coordinates \((X, Y)\) of a point \((t, x)\) we use the quantities

\[ X = \int_0^{x^-(0, t, x)} \left( 1 + R^2(0, x) \right) dx, \quad Y = \int_{x^+(0, t, x)}^0 \left( 1 + S^2(0, x) \right) dx \]

space + energy of backward waves  \quad space + energy of forward waves
Coordinate change: independent variables

\[ X_t - c(u)X_x = 0 \quad Y_t + c(u)Y_x = 0 \]

Differentiation rule for \( f = f(t, x) = f(X, Y) \)

\[
\begin{cases}
    f_t + cf_x &= 2cX_x f_X \\
    f_t - cf_x &= -2cY_x f_Y
\end{cases}
\]
Resolve the singularity $R, S \approx \infty$ by introducing the variables

$$\alpha \doteq 2 \arctan R \quad \beta \doteq 2 \arctan S$$

so that

$$R = \tan \frac{\alpha}{2} \quad S = \tan \frac{\beta}{2}$$

$$R, S \rightarrow \pm \infty \quad \iff \quad \alpha, \beta \rightarrow -\pi$$
Moreover define \[ p = \frac{1 + R^2}{X_x}, \quad q = \frac{1 + S^2}{-Y_x}, \] so that

\[ (X_x)^{-1} = \frac{p}{1 + R^2} = p \cos^2 \alpha, \quad (-Y_x)^{-1} = \frac{q}{1 + S^2} = q \cos^2 \beta \]
A semilinar system in characteristic variables

\[ u_{tt} - c(u)(c(u)u_x)_x = 0 \]

\[
\begin{align*}
\alpha_Y &= \frac{c'}{8c^2} (\cos \beta - \cos \alpha) q \\
\beta_X &= \frac{c'}{8c^2} (\cos \alpha - \cos \beta) p \\
p_Y &= \frac{c'}{8c^2} \left[ \sin \beta - \sin \alpha \right] pq \\
q_X &= \frac{c'}{8c^2} \left[ \sin \alpha - \sin \beta \right] pq 
\end{align*}
\]

\[
u_Y = \frac{\sin \beta}{4c} q \quad u_X = \frac{\sin \alpha}{4c} p.
\]

\[(t, x) \leftrightarrow (X, Y)\]
Boundary data

\[ u(0, x) = u_0(x) \quad u_t(0, x) = u_1(x) \]

\[(u_0)_x \in L^2, \quad u_1 \in L^2 \implies R, S \in L^2 \]
Along the curve $\gamma = \{ Y = \varphi(X) \}$ parametrized by $x \mapsto (X(x), Y(x))$, the boundary data $\left( \tilde{w}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{u} \right) \in L^\infty$ are defined by

\[
\begin{align*}
\tilde{\alpha} &= 2 \arctan R(0, x) \\
\tilde{\beta} &= 2 \arctan S(0, x)
\end{align*}
\]

\[
\begin{align*}
\tilde{p} &\equiv 1 \\
\tilde{q} &\equiv 1 \\
\tilde{u} &= u_0(x)
\end{align*}
\]

**Global solution:** the fixed point of a contractive transformation

\[
\begin{align*}
\alpha_Y &= \frac{c'}{8c^2} \left( \cos \beta - \cos \alpha \right) q \\
\beta_X &= \frac{c'}{8c^2} \left( \cos \alpha - \cos \beta \right) p \\
p_Y &= \frac{c'}{8c^2} \left[ \sin \beta - \sin \alpha \right] pq \\
q_X &= \frac{c'}{8c^2} \left[ \sin \alpha - \sin \beta \right] pq \\
u_Y &= \frac{\sin \beta}{4c} q, \\
u_X &= \frac{\sin \alpha}{4c} p, \\
c &= c(u)
\end{align*}
\]
Global existence of weak solutions

\[ u_{tt} + c(u)(c(u)u_x)_x = 0 \]
\[ u(0, x) = u_0(x) \quad u_t(0, x) = u_1(x) \]

**Theorem (A.B. - Y.Zheng, 2004).** Assume:

- \( c : \mathbb{R} \mapsto [a, b] \) smooth, with \( 0 < a < b \)

- \( u_0 \) absolutely continuous, \( (u_0)_x, u_1 \in L^2(\mathbb{R}) \)

**Existence** The Cauchy problem admits a global, Hölder continuous weak solution \( u = u(t, x) \).

\[ \int \int \left[ \phi_t u_t - (c(u)\phi)_x c(u) u_x \right] dxdt = 0 \quad \phi \in C^1_c \]
Continuous dependence

**w.r.t. Time:** The map \( t \mapsto u(t, \cdot) \) is Lipschitz continuous into \( L^2 \)

\[
\| u(t, \cdot) - u(s, \cdot) \|_{L^2} \leq L |t - s| \quad s, t \in \mathbb{R}
\]

and \( C^1 \) as a map with values in \( L^p_{\text{loc}} \) for \( p \in [1, 2[ \).

**w.r.t. Initial Data:** For a sequence of initial data

\[
\| (u^n_0)_x - (u_0)_x \|_{L^2} \to 0, \quad \| u^n_1 - u_1 \|_{L^2} \to 0,
\]

with \( u^n_0(x) \to u_0(x) \) uniformly on compact sets,

as \( n \to \infty \) one has the convergence

\[ u^n(t, x) \to u(t, x) \]

uniformly on bounded subsets of the \( t-x \) plane.
Conservation of energy

There exists a continuous family \( \{ \mu_t \}_{t \in \mathbb{R}} \) of positive Radon measures on the real line such that

(i) At every time \( t \), one has

\[
\mu_t(\mathbb{R}) = E_0 = \int \left[ u_t^2(x) + c^2(u_0(x))(u_0)_x^2(x) \right] dx
\]

(ii) For each \( t \), the absolutely continuous part of \( \mu_t \) has density \( u_t^2 + c^2 u_x^2 \) w.r.t. Lebesgue measure.

(iii) For almost every \( t \in \mathbb{R} \), the singular part of \( \mu_t \) is concentrated on the set where \( c'(u) = 0 \).
Energy transfer

\[
\begin{aligned}
(R^2)_t - (cR^2)_x &= \frac{c'}{2c} (R^2 S - RS^2) \\
(S^2)_t + (cS^2)_x &= -\frac{c'}{2c} (R^2 S - RS^2)
\end{aligned}
\]

\[Q(t) = \int \int_{x<y} R^2(y)S^2(x) \, dx \, dy = \text{interaction potential}\]

\[\chi(t) = \int 2c(u) R^2 S^2 \, dx = \text{instantaneous interaction}\]
\begin{align*}
\int \frac{c'}{c} |R^2 S - RS^2| \, dx &\leq C \cdot \int |R^2 S| + |S^2 R| \, dx \\
&\leq \frac{1}{2} \int R^2 (C_\varepsilon + \varepsilon S^2) + S^2 (C_\varepsilon + \varepsilon R^2) \, dx \leq C_\varepsilon E_0 + \varepsilon \int R^2 S^2 \, dx \\
\frac{d}{dt} Q(t) &\leq - \int 2c(u) R^2 S^2 \, dx + E_0 \int \frac{c'}{c} |R^2 S - RS^2| \, dx \\
&\leq C_\varepsilon E_0^2 + \int (\varepsilon E_0 - 2c(u)) R^2 S^2 \, dx \leq C_\varepsilon E_0^2 - \int \varepsilon E_0 R^2 S^2 \, dx \\
\int_0^T \int \varepsilon E_0 R^2 S^2 \, dx &\leq Q(0) - Q(T) + T \cdot C_\varepsilon E_0^2 \leq (1 + TC_\varepsilon) E_0^2 \\
\int_0^T \int \frac{c'}{c} |R^2 S - RS^2| \, dx dt &\leq TC_\varepsilon E_0 + (1 + TC_\varepsilon) E_0
\end{align*}

total amount of transferred energy increases linearly in time
Some open questions

- Dissipative solutions
- Blow-up patterns
- Characterization of limiting solutions
- Uniqueness of conservative solutions
Dissipative solutions

\[ u_{tt} - c(u)(c(u)u_x)_x = 0 \]

\[ \alpha = 2 \arctan R \]
\[ \beta = 2 \arctan S \]

\[ p = \frac{1 + R^2}{X_x}, \quad q = \frac{1 + S^2}{-Y_x} \]

\[ \begin{cases} 
\alpha_Y = \frac{c'}{8c^2} (\cos \beta - \cos \alpha) q, \\
\beta_X = \frac{c'}{8c^2} (\cos \alpha - \cos \beta) p, \\
p_Y = \frac{c'}{8c^2} \left[ \sin \beta - \sin \alpha \right] pq, \\
q_X = \frac{c'}{8c^2} \left[ \sin \alpha - \sin \beta \right] pq, 
\end{cases} \]

\[ u_Y = \frac{\sin \beta}{4c} q, \quad u_X = \frac{\sin \alpha}{4c} p. \]
Dissipative solutions for Hunter-Saxton

**Dissipative solution:** \( \int_{0}^{\infty} u_x^2 \, dx \) decreasing in time

**Conservative solution:** \( \int_{0}^{\infty} u_x^2 \, dx \) constant in time (possibly as a singular measure)
Toy model 3

Well posed: \[ (\dot{x}_1, \dot{x}_2) = \begin{cases} (1, -1) & \text{if } x_2 > 0 \\ (0, 0) & \text{if } x_2 = 0 \end{cases} \]

Ill posed: \[ (\dot{x}_1, \dot{x}_2) = \begin{cases} (1, x_1) & \text{if } x_2 > 0 \\ (0, 0) & \text{if } x_2 = 0 \end{cases} \]
\( c(u) \) monotone \( \rightarrow \) Cauchy problem is well posed?
$c(u)$ not monotone $\iff$ Cauchy problem is ill posed

or else

or else
Blow-up patterns for Burgers’ equation

Structurally unstable

Structurally stable
Structurally stable blow-up pattern for Hunter-Saxton

(For Camassa-Holm, or variational wave equation, the generic blow-up pattern should be the same)
Structurally stable blow-up pattern, in the dissipative case

\[ u(0) \]

\[ u(\tau) \]

\[ u(t) \]

\[ u_x = \infty \]
Weak limits of solutions - Burgers equation

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0 \]

Assume:

1. \( u_n : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \) is an entropy weak solution, \( n \geq 1 \)
2. weak convergence: \( u_n \rightharpoonup u \) as \( n \to \infty \)

Then \( u = u(t, x) \) is an entropy-weak solution as well

**Proof.** By Oleinik’s inequality, for every \( t > 0 \) the map \( x \mapsto u(t, x) \) is in \( \text{BV} \), hence taking a subsequence \( u_{n_j} \to u \) in \( \mathbb{L}^1_{loc} \)
\[ u_t + (u^2/2)_x = \frac{1}{2} \int_0^x u_x^2 \, dx \]

Assume:

- \( u_n = u_n(t, x) \) sequence of conservative solutions,
- \( u_n \to u \) uniformly on compact sets of \([0, T] \times \mathbb{R}_+\)

Question: is \( u \) also a solution?

If not, characterize all limits of solution sequences
Conjecture: the following are equivalent:

(1) $u = u(t, x)$ is a pointwise limit of solutions

(2) Then there exists a positive Radon measure $\mu^{excess}$ such that, introducing the interval

$$I(y) = \left[ \int_0^y u_x^2(0, x) \, dx + \mu^{excess}([0, y]), \quad \int_0^y u_x^2(0, x) \, dx + \mu^{excess}([0, y]) \right]$$

for every $t \geq 0$ the graph of $u(t)$ is described by

$$\text{Graph}(u(t)) = \left\{(x, w), \ w = u(0, y) + \lambda t, \ x = y + tu(0, y) + \lambda t^2/2 \right\}$$

for some $y \geq 0$, $\lambda \in I(y)$
More Conjectures:

- In general, that the uniform limit of solutions is not a solution.
- The above limit is a solution if only if $\mu^{excess}$ is purely atomic.

**Note:** similar results should hold for Camassa-Holm and for the variational wave equation.
Uniqueness of conservative solutions to the variational wave equation

\[ u_{tt} - c(u)(c(u)u_x)_x = 0 \]

Smooth solution: \((u, R, S)\)

Perturbed solutions:

\[ u^\varepsilon = u + \varepsilon v + o(\varepsilon), \quad R^\varepsilon = R + \varepsilon r + o(\varepsilon), \quad S^\varepsilon = S + \varepsilon s + o(\varepsilon) \]

\[ u^\varepsilon_t = \frac{R^\varepsilon + S^\varepsilon}{2} = \frac{R + S}{2} + \varepsilon \frac{r + s}{2} + o(\varepsilon), \]

\[ u^\varepsilon_x = \frac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)} = \frac{R - S}{2c(u)} + \varepsilon \frac{r - s}{2c(u)} - \varepsilon \frac{R - S}{4c^2(u)} c'(u) v + o(\varepsilon) \]

\[ v_x = - \frac{(R - S)c'(u)}{4c^2(u)} v + \frac{r - s}{2c(u)} \]
Evolution equations for first order perturbations $v, s, r$

\[
v_{tt} - c^2 v_{xx} = cc' v_x + \left( (c')^2 u_x + cc'' u_x + 2 cc' u_{xx} \right) v
\]

\[
\begin{align*}
  r_t - c(u) r_x & = c' R_x v + \left( \frac{c''}{4c} - \frac{(c')^2}{4c^2} \right) (R^2 - S^2) v + \frac{c'}{2c} (R r - S s) \\
  s_t + c(u) s_x & = - c' S_x v + \left( \frac{c''}{4c} - \frac{(c')^2}{4c^2} \right) (S^2 - R^2) v + \frac{c'}{2c} (S s - R r)
\end{align*}
\]
A weighted norm for tangent vectors

**Basic problem:** find a norm \( \| (v, r, s) \|_u \)

such that, for every solution \((u, R, S)\) and every solution \((v, r, s)\) of the corresponding linearized system, one should have

\[
\frac{d}{dt} \left\| (v(t), r(t), s(t)) \right\|_u(t) \leq C \cdot \left\| (v(t), r(t), s(t)) \right\|_u(t)
\]
Choosing $\|v\|_u = \|v\|_{H^1}$ yields the norm distance in $H^1$. This cannot work.

Next attempt: allow shifts in components $R, S$.
\[ \left\| (v, r, s) \right\|_u = \inf_{\tilde{r} - wR_x = r, \tilde{s} - zS_x = s} \left\| (\tilde{r}, w, \tilde{s}, z) \right\|_u, \]
A weighted infinitesimal distance

\[ \left\| (\tilde{r}, w, \tilde{s}, z) \right\|_u \]
\[ = \int_{-1}^{1} \left\{ [\text{change in } u] + [\text{change in } x] + [\text{change in } 2 \arctan R] \right\} (1 + \kappa W^-)(1 + R^2) \, dx \]
\[ + \int_{-1}^{1} [\text{change the base measure } \mu \text{ with density } 1 + R^2](1 + \kappa W^-) \, dx \]
\[ + \int_{-1}^{1} \left\{ [\text{change in } u] + [\text{change in } x] + [\text{change in } 2 \arctan S] \right\} (1 + \kappa W^+)(1 + S^2) \, dx \]
\[ + \int_{-1}^{1} [\text{change the base measure } \nu \text{ with density } 1 + S^2](1 + \kappa W^+) \, dx \]

Weights

\[ W^+(x) = \int_{x-1}^{x+1} (1 + x - y) R^2(y) \, dy , \quad W^-(x) = \int_{x-1}^{x} (1 + y - x) S^2(y) \, dy \]
More precisely, observing that

\[ \|v\|_{L^\infty} = \mathcal{O}(1) \cdot \int_{-1}^{1} |v_x| \, dx = \mathcal{O}(1) \cdot \int_{-1}^{1} (|r| + |s|) \, dx \]

we guess

\[ \left\| (\tilde{r}, w, \tilde{s}, z) \right\|_u \]

\[ = \int_{-1}^{1} \left\{ |\tilde{r} - wR_x| + |\tilde{s} - zS_x| \right\} \, dx \]

\[ + \int_{-1}^{1} \left\{ |w|(1 + \kappa W^-)(1 + R^2) + |z|(1 + \kappa W^+)(1 + S^2) \right\} \, dx \]

\[ + \int_{-1}^{1} \left\{ |\tilde{r}|(1 + \kappa W^-) + |\tilde{s}|(1 + \kappa W^+) \right\} \, dx \]

\[ + \int_{-1}^{1} \left\{ 2R\tilde{r} + (1 + R^2)w_x| + 2S\tilde{s} + (1 + S^2)z_x| \right\} \, dx \]
Missing: source terms generated by crossing

\[
\begin{aligned}
R_t - cR_x &= \frac{c'}{4c}(R^2 - S^2) \\
S_t + cS_x &= \frac{c'}{4c}(S^2 - R^2)
\end{aligned}
\]

\[
\begin{aligned}
r &= \bar{r} - wR_x + \frac{c'}{8c^2}(R^2 - S^2)(w - z) \\
s &= \bar{s} - zS_x + \frac{c'}{8c^2}(R^2 - S^2)(z - w)
\end{aligned}
\]
Shift components $w, z$ can be propagated along characteristics 

Choosing $w(0, \cdot), z(0, \cdot)$ at time $t = 0$ yields a natural choice for $w(t, \cdot), z(t, \cdot)$ at every $t > 0$

\[
\begin{align*}
    w_t - c(u)w_x &= -c'(u)(v + u_x w) \\
    z_t + c(u)z_x &= c'(u)(v + u_x z)
\end{align*}
\]

Guess: \[
\frac{d}{dt} \left\| (v(t), r(t), s(t)) \right\|_{u(t)} \leq C \cdot \left\| (v(t), r(t), s(t)) \right\|_{u(t)}
\]