1 Elliptic equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given measurable functions $a^{ij}, b^i, c : \Omega \rightarrow \mathbb{R}$, consider the linear, second order differential operator

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n (b^i(x)u)_{x_i} + c(x)u.$$  \hspace{1cm} (1.1)

In this chapter, we study solutions to the boundary value problem

$$\begin{cases}
Lu = f & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}$$  \hspace{1cm} (1.2)

where $f \in L^2(\Omega)$ is a given function.

For future reference, we collect the main assumptions used throughout this chapter.

\textbf{(H)} The domain $\Omega \subset \mathbb{R}^n$ is open and bounded. The coefficients of $L$ in (1.1) satisfy

$$a^{ij}, b^i, c \in L^\infty(\Omega).$$  \hspace{1cm} (1.3)

Moreover, the operator $L$ is \textbf{uniformly elliptic}. Namely, there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega, \ \xi \in \mathbb{R}^n.$$  \hspace{1cm} (1.4)

\textbf{Remark 1.1} By definition, the uniform ellipticity of the operator $L$ depends only on the coefficients $a^{ij}$. In the symmetric case where $a^{ij} = a^{ji}$, the above condition means that for every $x \in \Omega$ the $n \times n$ symmetric matrix $A(x) = (a^{ij}(x))$ is strictly positive definite, and its smallest eigenvalue is $\geq \theta$.

1.1 Physical interpretation

As an example, consider a fluid moving with velocity $b(x) = (b^1, b^2, b^3)(x)$ in a domain $\Omega \subset \mathbb{R}^3$ Let $u = u(t, x)$ describe the density of a chemical dispersed within the fluid.
Given any subdomain $V \subset \Omega$, we assume that the total amount of chemical contained in $V$ changes only due to the inward or outward flow through the boundary $\partial V$. Namely:

$$\frac{d}{dt} \int_V u \, dx = \int_{\partial V} n \cdot (a \nabla u) \, dS - \int_{\partial V} n \cdot (b u) \, dS. \quad (1.5)$$

Here $n(x)$ denotes the unit outer normal to the set $V$ at a boundary point $x$, while $a > 0$ is a constant diffusion coefficient. The first integral on the right hand side of (1.5) describes how much chemical enters through the boundary by \textit{diffusion}. Notice that this is positive if $n \cdot \nabla u > 0$. Roughly speaking, this is the case if the concentration of chemical outside the domain $V$ is greater than inside. The second integral (with the minus sign in front) denotes the amount of chemical that moves out across the boundary of $V$ by \textit{advection}, being transported by the fluid in motion (Fig. 1).

![Figure 1: As the chemical is transported across the boundary of $V$, the total amount contained inside $V$ changes in time.](image)

Using the divergence theorem, from (1.5) we obtain

$$\int_V u_t \, dx = \int_V a \Delta u \, dx - \int_V \text{div}(b u) \, dx. \quad (1.6)$$

Since the above identity holds on every subdomain $V \subset \Omega$, we conclude that $u = u(t, x)$ satisfies the parabolic PDE

$$u_t - a \Delta u + \text{div}(b u) = 0. \quad [\text{diffusion}]$$

$$[\text{advection}]$$

A more general model can describe the following situations:

- The diffusion is not uniform throughout the domain. In other words, the coefficient $a$ is not a constant but depends on the location $x \in \Omega$. Moreover, the diffusion is not isotropic: in some directions it is faster than in others. All this can be modeled by replacing the constant diffusion matrix $A(x) \equiv aI$ with a more general symmetric matrix $A(x) = (a^{ij}(x))$.

- The total amount of chemical is not conserved. Additional terms are present, accounting for linear decay and for an external source.

In $n$ space dimensions, this leads to a linear evolution equation of the form

$$u_t - \sum_{i,j=1}^{n} \left( a^{ij}(x) u_{x_i} \right)_{x_j} + \sum_{i=1}^{n} \left( b^i(x) u \right)_{x_i} = -c(x) u + f(x). \quad (1.7)$$

$$[\text{diffusion}]$$

$$[\text{advection}]$$

$$[\text{decay}]$$

$$[\text{source}]$$
The equation (1.7) can be used to model a variety of phenomena, including mass transport, heat propagation, etc. In many situations, one is interested in **steady states**, i.e. in solutions which are independent of time. Setting $u_t = 0$ in (1.7) we obtain the linear elliptic equation

\[- \sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_x_j + \sum_{i=1}^{n} (b^i(x)u)_x_i + c(x)u = f(x). \] (1.8)

### 1.2 Weak solutions

By a **classical solution** of the boundary value problem (1.2) we mean a function $u \in C^2(\Omega)$ which satisfies the equation and the boundary conditions at every point. In general, due to the lack of regularity in the coefficients of the equation, we do not expect that (1.2) will have classical solutions. A weaker concept of solution is thus needed.

**Definition 1.1** A **weak solution** of (1.2) is a function $u \in H^1_0(\Omega)$ such that

\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} - \sum_{i=1}^{n} b^i u v_{x_i} + c u v \right) dx = \int_{\Omega} f v \, dx \quad \text{for all } v \in H^1_0(\Omega). \] (1.9)

**Remark 1.2** The above equality is formally obtained by writing

\[ \int_{\Omega} (Lu) v \, dx = \int_{\Omega} f v \, dx \quad \text{for all } v \in C_c^\infty(\Omega), \] (1.10)

and integrating by parts. Notice that, if (1.10) holds for all test functions $v \in C_c^\infty(\Omega)$, then by an approximation argument the same integral identity is valid for all $v \in H^1_0(\Omega)$. We observe that a function $u \in H^1_0$ may not have weak derivatives of second order. However, the integral in (1.9) is always well defined, for all $u, v \in H^1_0$.

A convenient way to reformulate the concept of weak solution is the following. On the Hilbert space $H^1_0(\Omega)$, consider the bilinear form

\[ B[u, v] = \int_{\Omega} \left( \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} - \sum_{i=1}^{n} b^i u v_{x_i} + c u v \right) dx. \] (1.11)

A function $u \in H^1_0$ is a weak solution of (1.2) provided that

\[ B[u, v] = (f, v)_{L^2} \quad \text{for all } v \in H^1_0. \] (1.12)

Here and in the sequel we use the notation

\[ (f, g)_{L^2} = \int_{\Omega} f g \, dx \] (1.13)

for the inner product in $L^2(\Omega)$, to distinguish it from the inner product in $H^1(\Omega)$

\[ (f, g)_{H^1} = \int_{\Omega} f g \, dx + \int_{\Omega} \sum_{i=1}^{n} f_{x_i} g_{x_i} \, dx. \] (1.14)
Remark 1.3 The minus sign in front of the second order terms in (1.1) disappears in (1.11), after a formal integration by parts. As it will become apparent later, this sign is chosen so that the corresponding quadratic form $B[u, v]$ can be positive definite.

Remark 1.4 (general boundary conditions). The homogeneous boundary condition $u = 0$ on $\partial \Omega$ is incorporated in the requirement $u \in H_{0}^{1}(\Omega)$. More generally, given a function $g \in H^{1}(\Omega)$, one can consider the non-homogeneous boundary value problem

$$\begin{cases} Lu = f & x \in \Omega, \\ u = g & x \in \partial \Omega. \end{cases}$$

(1.15)

This can be rewritten as a homogeneous problem for the function $\tilde{u} = u - g$, namely

$$\begin{cases} L\tilde{u} = f - Lg & x \in \Omega, \\ \tilde{u} = 0 & x \in \partial \Omega. \end{cases}$$

(1.16)

Assuming that $Lg \in L^{2}$, the problem (1.16) is exactly of the same type as (1.2).

Remark 1.5 (operators not in divergence form). A differential operator of the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_{i}x_{j}} + \sum_{i=1}^{n} b^{i}(x)u_{x_{i}} + c(x)u$$

can be rewritten as

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_{i}x_{j}})_{x_{j}} + \sum_{i=1}^{n} (b^{i}(x) + \sum_{j=1}^{n} a^{ij}_{x_{j}}(x))u_{x_{i}} + c(x)u.$$

Assuming that $a^{ij}, a^{ij}_{x_{j}}, b^{i}, c \in L^{\infty}(\Omega)$, a weak solution of the corresponding problem (1.2) can be again obtained by solving (1.12), where the bilinear form $B$ is now defined by

$$B[u, v] = \int_{\Omega} \left( \sum_{i,j=1}^{n} a^{ij}_{x_{i}x_{j}}u_{x_{i}x_{j}}v + \sum_{i=1}^{n} (b^{i}(x) + \sum_{j=1}^{n} a^{ij}_{x_{j}}(x))u_{x_{i}}v + uv \right) dx.$$

As a first example, consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & x \in \Omega, \\ u = 0 & x \in \partial \Omega. \end{cases}$$

(1.17)

Clearly, the operator $-\Delta u = -\sum_{i} u_{x_{i}x_{i}}$ is uniformly elliptic, because in this case the $n \times n$ matrix $A(x) = (a^{ij}(x))$ is the identity matrix, for every $x \in \Omega$. The existence of solutions to (1.17) is provided by

Lemma 1.1 Let $\Omega \subset \mathbb{R}^{n}$ be a bounded open set. Then for every $f \in L^{2}(\Omega)$ the boundary value problem (1.17) has a unique weak solution $u \in H_{0}^{1}(\Omega)$. The corresponding map $f \mapsto u$ is a compact linear operator from $L^{2}(\Omega)$ into $H_{0}^{1}(\Omega)$. 

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Proof. By the Rellich-Kondrachov theorem, the canonical embedding $\iota : H^1_0(\Omega) \to L^2(\Omega)$ is compact. Hence its dual operator $\iota^*$ is also compact. Since $H^1_0$ and $L^2$ are Hilbert spaces, they can be identified with their duals. We thus obtain the following diagram:

$$H^1_0(\Omega) \xrightarrow{\iota} L^2(\Omega) \quad (1.18)$$

For each $f \in L^2(\Omega)$, the definition of dual operator yields

$$(\iota^* f, v)_{H^1} = (f, \iota v)_{L^2} = (f, v)_{L^2}$$

for all $f \in L^2(\Omega), v \in H^1_0(\Omega)$.

By (1.14) this means that $\iota^* f$ is precisely the weak solution to (1.17).

1.3 Existence and uniqueness of solutions

We begin by studying solutions to the elliptic boundary value problem

$$\begin{cases}
Lu = f & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases} \quad (1.19)$$

assuming that the differential operator $L$ contains only second order terms:

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j}. \quad (1.20)$$

We recall that a weak solution of (1.19) is a function $u \in H^1_0(\Omega)$ such that

$$B[u, v] = (f, v)_{L^2} = (\iota^* f, v)_{H^1}$$

for all $v \in H^1_0(\Omega)$, where $B : H^1_0 \times H^1_0 \to \mathbb{R}$ is the continuous bilinear form

$$B[u, v] = \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} \, dx. \quad (1.22)$$

Theorem 1.1 (solution of a linear elliptic boundary value problem - I). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the operator $L$ in (1.20) be uniformly elliptic, with coefficients $a^{ij} \in L^\infty(\Omega)$. Then, for every $f \in L^2(\Omega)$, the boundary value problem (1.19) has a unique weak solution $u \in H^1_0(\Omega)$. The corresponding solution operator, which we denote as $L^{-1} : f \mapsto u$ is a compact linear operator from $L^2(\Omega)$ into $H^1_0(\Omega)$.

Proof. The existence and uniqueness of weak solutions to the elliptic boundary value problem (1.19) will be achieved by checking that the bilinear form $B_0$ satisfies all the assumptions of the Lax-Milgram theorem.

1. The continuity of $B$ is clear. Indeed,

$$|B[u, v]| \leq \sum_{i,j=1}^n \int_\Omega |a^{ij} u_{x_i} v_{x_j}| \, dx \leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty} \|u_{x_i}\|_{L^2} \|v_{x_j}\|_{L^2} \leq C \|u\|_{H^1} \|v\|_{H^1}.$$

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2. We claim that $B$ is strictly positive definite, i.e. there exists $\beta > 0$ such that

$$B[u, u] \geq \beta \|u\|_{H^1(\Omega)}^2$$

for all $u \in H^1_0(\Omega)$. \hfill (1.23)

Indeed, since $\Omega$ is bounded, Poincaré’s inequality yields the existence of a constant $\kappa$ such that

$$\|u\|_{L^2(\Omega)}^2 \leq \kappa \int_\Omega |\nabla u|^2 \, dx$$

for all $u \in H^1_0(\Omega)$.

On the other hand, the uniform ellipticity condition implies

$$B[u, u] = \int_\Omega \sum_{i,j=1}^n a^{ij} u_x^i u_x^j \, dx \geq \int_\Omega \theta \sum_{i=1}^n u_{x_i}^2 \, dx = \theta \int_\Omega |\nabla u|^2 \, dx.$$

Together, the two above inequalities yield

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (\kappa + 1) \|\nabla u\|_{L^2}^2 \leq \frac{\kappa + 1}{\theta} B[u, u].$$

This proves (1.23) with $\beta = \theta/(\kappa + 1)$.

3. By the Lax-Milgram theorem, for every $\tilde{f} \in H^1_0(\Omega)$ there exists a unique element $u \in H^1_0$ such that

$$B[u, v] = (\tilde{f}, v)_{H^1}$$

for all $v \in H^1_0(\Omega).$ \hfill (1.24)

Moreover, the map $\Lambda : \tilde{f} \mapsto u$ is continuous, namely

$$\|u\|_{H^1} \leq \beta^{-1} \|\tilde{f}\|_{H^1}.$$

Choosing $\tilde{f} = \iota^* f \in H^1_0(\Omega)$, defined at (1.18), we thus achieve

$$B[u, v] = (\iota^* f, v)_{H^1} = (f, v)_{L^2}$$

for all $v \in H^1_0(\Omega).$ \hfill (1.25)

By definition, $u$ is a weak solution of (1.19).

4. To prove that the solution operator $L^{-1} : f \mapsto u$ is a compact, consider the the diagram

$$L^2(\Omega) \xrightarrow{\iota^*} H^1_0(\Omega) \xrightarrow{\Lambda} H^1_0(\Omega).$$

By Lemma 1.1, the linear operator $\iota^*$ is compact. Moreover, $\Lambda$ is continuous. Therefore the composition $L^{-1} = \Lambda \circ \iota^*$ is compact.

1.4 Representation of solutions

We now study in more detail the structure of the solution operator $L^{-1}$ for the elliptic boundary value problem (1.19)-(1.20).

**Theorem 1.2 (representation of solutions in terms of eigenfunctions).** Assume $a^{ij} = a^{ji} \in L^\infty(\Omega)$. Then, in the setting of Theorem 1.1, the linear operator $L^{-1} : L^2(\Omega) \mapsto L^2(\Omega)$ is compact, one-to-one, and self-adjoint.
The space $L^2(\Omega)$ admits an orthonormal basis $\{\phi_k; \ k \geq 1\}$ consisting of eigenfunctions of $L^{-1}$, and one has the representation

$$L^{-1} f = \sum_{k=1}^{\infty} \lambda_k (f, \phi_k)_{L^2} \phi_k. \quad (1.26)$$

The corresponding eigenvalues $\lambda_k$ satisfy

$$\lim_{k \to \infty} \lambda_k = 0, \quad \lambda_k > 0 \quad \text{for all} \quad k \geq 1. \quad (1.27)$$

**Proof.**

1. By Theorem 1.1, $L^{-1}$ is a compact linear operator from $L^2(\Omega)$ into itself.

2. If $u = L^{-1} f = 0$, then

$$0 = B[u, v] = (f, v)_{L^2}$$

for every $v \in H^1_0(\Omega)$. In particular, for every test function $\phi \in C_\infty^\infty(\Omega)$ we have

$$\int_\Omega f \phi \, dx = 0.$$

This implies $f(x) = 0$ for a.e. $x \in \Omega$. Hence $\text{Ker}(L^{-1}) = \{0\}$ and the operator $L^{-1}$ is one-to-one.

3. To prove that $L^{-1}$ is self-adjoint, assume

$$f, g \in L^2(\Omega), \quad u = L^{-1} f, \quad v = L^{-1} g.$$

Recalling the definition of weak solution at (1.9), this implies $u, v \in H^1_0(\Omega)$ and

$$(L^{-1} f, g)_{L^2} = \int_\Omega u g \, dx = \int_\Omega \sum_{i,j} a^{ij} u_x^i v_x^j \, dx = \int_\Omega f v \, dx = (f, L^{-1} g)_{L^2}.$$ 

Note that the second equality follows from the fact the $v = L^{-1} g$, using $u \in H^1_0(\Omega)$ as a test function. The third equality follows from the fact the $u = L^{-1} f$, using $v \in H^1_0(\Omega)$ as a test function.

4. By the previous steps, we can now apply Theorem 1 (Hilbert-Schmidt), providing the existence of an orthonormal basis of eigenfunctions of $L^{-1}$. This yields (1.26).

By the compactness of the operator $L^{-1}$, the eigenvalues satisfy $\lambda_k \to 0$. Finally, choosing $v = \phi_k$ as test function in (1.21), one obtains

$$B[\lambda_k \phi_k, \phi_k] = (\phi_k, \phi_k)_{L^2} = 1.$$ 

Since the quadratic form $B$ is strictly positive definite, we conclude

$$\lambda_k = \frac{1}{B[\phi_k, \phi_k]} > 0.$$
Example 1. Let $\Omega = ]0, \pi[ \subset \mathbb{R}$ and $Lu = -u_{xx}$. Given $f \in \mathbb{L}^2([0, \pi])$, consider the elliptic boundary value problem
\[
\begin{aligned}
- u_{xx} &= f, & 0 < x < \pi, \\
u(0) &= u(\pi) = 0.
\end{aligned}
\] (1.28)

As a first step, we compute the eigenfunctions of $L_0$. Solving the boundary value problem
\[- u_{xx} = \mu u, \quad u(0) = u(\pi) = 0.
we find the eigenvalues and the normalized eigenfunctions
\[
\mu_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx.
\]

Of course, the inverse operator $L^{-1}$ has the same eigenfunctions $\phi_k$, with eigenvalues $\lambda_k = 1/k^2$.

In this special case, the formula (1.26) yields the well known representation of solutions of (1.28) in terms of a Fourier sine series:
\[
u(x) = L^{-1} f = \sum_{k=1}^{\infty} \lambda_k (f, \phi_k) \mathbb{L}^2 \phi_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \left( \int_0^\pi f(y) \sqrt{\frac{2}{\pi}} \sin ky \, dy \right) \sqrt{\frac{2}{\pi}} \sin kx
= \sum_{k=1}^{\infty} \frac{2}{\pi k^2} \left( \int_0^\pi f(y) \sin ky \, dy \right) \sin kx.
\]

1.5 General linear elliptic operators

The existence and uniqueness result stated in Theorem 1.1 relied on the fact that, by Poincaré’s inequality, the bilinear form $B$ in (1.21) is strictly positive definite on the space $H^1_0(\Omega)$. These properties no longer hold for the more general bilinear form $B$ in (1.11). For example, if the function $c(x)$ is large and negative, one may find some $u \in H^1_0(\Omega)$ such that $B[u, u] < 0$.

Example 2. Consider the one-dimensional open interval $\Omega = ]0, \pi[$. The operator
\[
Lu = - u_{xx} - 4u
\] is uniformly elliptic on $\Omega$. However, the corresponding bilinear form
\[
B[u, v] = \int_0^\pi u_x v_x - 4uv \, dx
\] is not positive definite on $H^1_0(\Omega)$. For example, taking $u(x) = \sin x$ we find
\[
B[u, u] = \int_0^\pi \cos^2 x - 4 \sin^2 x \, dx = - \frac{3\pi}{2}.
\]

Observe that, if we take $f(x) = \sin 2x$, then the boundary value problem
\[
\begin{aligned}
- u_{xx} - 4u &= \sin 2x, & x \in ]0, \pi[, \\
u(0) &= u(\pi) = 0,
\end{aligned}
\] (1.29)
has no solutions. Indeed, taking \( v(x) = \sin 2x \), for every \( u \in H_0^1(\Omega) \) an integration by parts yields

\[
B[u, v] = \int_0^\pi u_x v_x - 4uv \, dx = \int_0^\pi \left( 2u_x \cos 2x - 4u \sin 2x \right) \, dx = 0 \neq \int_0^\pi \sin^2 2x \, dx .
\]

Therefore, it is not possible to satisfy the identity (1.12) for all \( v \in H_0^1(\Omega) \). In this connection, one should also observe that the corresponding homogeneous problem

\[
\left\{ \begin{array}{ll}
- u_{xx} - 4u &= 0 \\
u(0) &= u(\pi) = 0
\end{array} \right., \quad x \in ]0, \pi[, \tag{1.30}
\]

admits infinitely many solutions: \( u(x) = \kappa \sin 2x \), where \( \kappa \in \mathbb{R} \) is any constant.

We now study the existence and uniqueness of weak solutions to the more general boundary value problem (1.2). Before stating a precise result, we outline the two main steps in the approach.

**STEP 1:** By choosing a constant \( \gamma > 0 \) sufficiently large, the operator

\[
L_\gamma u = Lu + \gamma u \tag{1.31}
\]

is strictly positive definite. More precisely, the corresponding bilinear form

\[
B_\gamma[u, v] = \int_\Omega \left( \sum_{i,j=1}^n a^{ij}(x) u_{x_i} v_{x_j} - \sum_{i=1}^n b^i(x) u w_{x_i} + c(x) uv + \gamma uv \right) \, dx . \tag{1.32}
\]

satisfies

\[
B_\gamma[u, u] \geq \beta \| u \|^2_{H^1} \quad \text{for all } u \in H^1_0(\Omega), \tag{1.33}
\]

for some constant \( \beta > 0 \). Using the Lax-Milgram theorem, we conclude that for every \( f \in L^2(\Omega) \) the elliptic equation

\[
L_\gamma u = f
\]

has a unique solution \( u \in H^1_0(\Omega) \). Moreover, the map \( f \mapsto u = L_\gamma^{-1} f \) is a linear compact operator from \( L^2(\Omega) \) into \( H^1_0(\Omega) \subset L^2(\Omega) \). We regard \( L_\gamma^{-1} \) as a compact operator from \( L^2(\Omega) \) into itself.

**STEP 2:** The original problem (1.2) can now be written as

\[
Lu = L_\gamma u - \gamma u = f .
\]

Applying the operator \( L_\gamma^{-1} \) to both sides, one obtains

\[
u - L_\gamma^{-1} \gamma u = L_\gamma^{-1} f . \tag{1.34}
\]

Introducing the notation

\[
K = \gamma L_\gamma^{-1}, \quad h = L_\gamma^{-1} f , \tag{1.35}
\]

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we are led to the equation

$$(I - K)u = h.$$ (1.36)

Since $K$ is a compact operator from $L^2(\Omega)$ into itself, the Fredholm theory can be applied. In particular, one has

**Fredholm's alternative:** either

(i) for every $h \in L^2(\Omega)$ the equation $u - Ku = h$ has a unique solution $u \in L^2(\Omega)$,

or else

(ii) the equation $u - Ku = 0$ has a nontrivial solution $u \in L^2(\Omega)$.

Calling $K^* : L^2 \mapsto L^2$ the adjoint operator, case (ii) occurs if and only if the adjoint equation $v - K^*v = 0$ has a nontrivial solution $v \in L^2(\Omega)$. Information about the existence and uniqueness of weak solutions to (1.2) can thus be obtained by studying the adjoint operator

$$L^* v = - \sum_{i,j=1}^n (a^{ij}(x)v_{x_j})_{x_i} - \sum_{i=1}^n b^i(x)v_{x_i} + c(x)v.$$ (1.37)

In the remainder of this section we shall provide detailed proofs of the above claims.

**Lemma 1.2 (estimates on elliptic operators).** Let the operator $L$ in (1.1) be uniformly elliptic, with coefficients $a^{ij}, b^i, c \in L^\infty(\Omega)$. Then there exist constants $\alpha, \beta, \gamma > 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{H^1}\|v\|_{H^1},$$

$$\beta\|u\|^2_{H^1} \leq B[u,u] + \gamma \|u\|^2_{L^2},$$

for all $u, v \in H^1_0(\Omega)$.

**Proof.** 1. The boundedness of the bilinear form $B : H^1_0 \times H^1_0 \mapsto \mathbb{R}$ follows from

$$|B[u,v]| = \left| \int_{\Omega} \left( \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} - \sum_{i=1}^n b^i(u)v_{x_i} + cuv \right) dx \right|$$

$$\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty}\|u_{x_i}\|_{L^2}\|v_{x_j}\|_{L^2} + \sum_{i=1}^n \|b^i\|_{L^\infty}\|u\|_{L^2}\|v_{x_i}\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}$$

$$\leq \alpha \|u\|_{H^1}\|v\|_{H^1}.$$  

2. Toward the second estimate, using the ellipticity condition (1.4) and the elementary in-
equality \( ab \leq \frac{1}{2\theta}a^2 + \frac{\theta}{2}b^2 \), we obtain

\[
\theta \sum_{i=1}^{n} \|u_{x_i}\|^2_{L^2} \leq \int_{\Omega} \sum_{i=1}^{n} a^{ij}u_{x_i}u_{x_j} \, dx
\]

\[
= B[u, u] + \int_{\Omega} \left( \sum_{i,j=1}^{n} b^{ij}u_{x_i} - cu \right) \, dx
\]

\[
\leq B[u, u] + \sum_{i=1}^{n} \|b_i\|_{L^\infty} \|u\|_{L^2} \|u_{x_i}\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2
\]

\[
\leq B[u, u] + \left( \frac{1}{2\theta} \sum_{i=1}^{n} \|b_i\|_{L^\infty} \|u\|_{L^2} \|u_{x_i}\|_{L^2} + \frac{\theta}{2} \sum_{i=1}^{n} \|u_{x_i}\|_{L^2}^2 \right) + \|c\|_{L^\infty} \|u\|_{L^2}^2.
\]

Therefore

\[
B[u, u] \geq \frac{\theta}{2} \sum_{i=1}^{n} \|u_{x_i}\|_{L^2}^2 - C\|u\|_{L^2}^2 \quad \text{for all} \ u \in H_0^1(\Omega)
\]

for a suitable constant \( C \). Taking \( \beta = \theta/2 \) and \( \gamma = C + \theta/2 \), the inequality (1.39) is satisfied.

**Remark 1.6** By the above lemma, by choosing the constant \( \gamma > 0 \) large enough, the bilinear form \( B_\gamma \) in (1.32) is strictly positive definite. Notice that, for \( \gamma > 0 \) large, it would be very easy to show that the bilinear form

\[
\tilde{B}_\gamma = B[u, v] + \gamma(u, v)_{H^1}
\]

is strictly positive definite on \( H_0^1(\Omega) \). However, Lemma 1.2 shows that we can achieve strict positivity by adding the much weaker term \( \gamma(u, v)_{L^2} \).

Let \( \gamma \) be as in (1.39) and define the bilinear form \( B_\gamma \) according to (1.32). Since \( B_\gamma \) is positive definite, we can apply the Lax-Milgram theorem and conclude that, for every \( f \in L^2(\Omega) \), there exists a unique \( u \in H_0^1(\Omega) \) such that

\[
B_\gamma[u, v] = (f, v)_{L^2} = (i^* f, v)_{H^1} \quad \text{for all} \ v \in H_0^1(\Omega).
\]

(1.40)

Since the map \( i^* \) is compact, the solution operator \( f \mapsto u = L^{-1}_\gamma f \) is a linear compact operator from \( L^2(\Omega) \) into \( H_0^1(\Omega) \). Therefore, it is also a compact operator from \( L^2(\Omega) \) into itself.

An entirely similar result holds for the adjoint problem

\[
\begin{aligned}
L^*_\gamma v &= g & x \in \Omega, \\
v &= 0 & x \in \partial\Omega,
\end{aligned}
\]

(1.41)

where \( L^* \) is the adjoint operator introduced in (1.37), and \( L^*_\gamma u = L^* u + \gamma u \). Given \( g \in L^2(\Omega) \), a weak solution of (1.41) is defined to be a function \( v \in H_0^1(\Omega) \) such that

\[
B^*_\gamma[v, u] = B_\gamma[u, v] = (u, g)_{L^2} \quad \text{for all} \ u \in H_0^1(\Omega).
\]

(1.42)

Since \( B^*_\gamma \) is strictly positive definite, for every \( g \in L^2 \) the Lax-Milgram theorem yields a unique weak solution \( v \) of (1.41). The map \( g \mapsto v = (L^*_\gamma)^{-1} g \) is a linear, compact operator from \( L^2(\Omega) \) into itself.
Lemma 1.3 (adjoint operator). In the above setting, the operator $(L^*_\gamma)^{-1}$ is the adjoint of the operator $L^{-1}_\gamma$.

**Proof.** By definition, for every $f, g \in L^2(\Omega)$ and $u, v \in H^1_0(\Omega)$ one has
\[
(f,v)_{L^2} = B_\gamma [L^{-1}_\gamma f, v], \quad (u,g)_{L^2} = B_\gamma [u, (L^{-1}_\gamma)^* g].
\]
In particular, choosing $v = (L^*_\gamma)^{-1} g$ and $u = L^{-1}_\gamma f$ we obtain
\[
(f, (L^*_\gamma)^{-1} g)_{L^2} = B_\gamma [L^{-1}_\gamma f, (L^{-1}_\gamma)^* g] = (L^{-1}_\gamma f, g)_{L^2}.
\]

Lemma 1.4 (representation of weak solutions). Given any $f \in L^2(\Omega)$, a function $u \in L^2(\Omega)$ is a weak solution of (1.2) if and only if
\[
(I - K)u = h, \quad \text{with} \quad K = \gamma L^{-1}_\gamma, \quad h = L^{-1}_\gamma f.
\]

**Proof.** 1. Let $u$ be a weak solution of (1.2). By definition of weak solution, this implies $u \in H^1_0(\Omega)$ and
\[
B_\gamma [u,v] = B[u,v] + \gamma (u,v)_{L^2} = (f + \gamma u,v)_{L^2} \quad \text{for all} \quad v \in H^1_0(\Omega).
\]
Therefore
\[
u = L^{-1}_\gamma (f + \gamma u) = h + Ku.
\]

2. To prove the converse, let (1.43) hold. Then
\[
u = \gamma L^{-1}_\gamma u + L^{-1}_\gamma f \in H^1_0(\Omega).
\]
Moreover, for every $v \in H^1_0(\Omega)$ we have
\[
B[u,v] = B_\gamma [u,v] - \gamma (u,v)_{L^2} = (f + \gamma u,v)_{L^2} - \gamma (u,v)_{L^2} = (f,v)_{L^2}.
\]

In order to apply Fredholm’s theory, together with (1.2) we consider the homogeneous problem
\[
\begin{aligned}
Lu & = 0 & x \in \Omega, \\
u & = 0 & x \in \partial \Omega,
\end{aligned}
\]
and the adjoint problem
\[
\begin{aligned}
L^* v & = 0 & x \in \Omega, \\
v & = 0 & x \in \partial \Omega,
\end{aligned}
\]
where $L^*$ is the adjoint linear operator defined at (1.37).
Theorem 1.3 (solutions to the elliptic boundary value problem - II). Under the basic assumptions (H), the following statements are equivalent:

(i) For every \( f \in L^2(\Omega) \), the elliptic boundary value problem (1.2) has a unique weak solution.

(ii) The homogeneous boundary value problem (1.44) has the only solution \( u(x) \equiv 0 \).

(iii) The adjoint homogeneous problem (1.45) has the only solution \( v(x) \equiv 0 \).

Proof. 1. Since \( K = \gamma L^{-1} \gamma \) is a compact operator from \( L^2(\Omega) \) into itself, Fredholm’s theorem can be applied. As a consequence, the linear operator \( I - K \) is surjective if and only if it is one-to-one, i.e. if and only if \( \text{Ker}(I - K) = \{0\} \).

2. By Lemma 1.4, \( u - Ku = 0 \) if and only if \( u \) is a weak solution of (1.44). An entirely similar argument shows that \( v - K^*v = 0 \) if and only if \( v \) is a weak solution of (1.45).

By Fredholm’s theorem, \( \text{Ker}(I - K) \) and \( \text{Ker}(I - K^*) \) have the same dimension. We thus obtain a chain of equivalent statements:

\[
I - K \text{ is surjective } \iff \text{Ker}(I - K) = \{0\} \iff \text{Ker}(I - K^*) = \{0\} \iff u \equiv 0 \text{ is the unique solution of (1.44)} \iff v \equiv 0 \text{ is the unique solution of (1.45)}.
\]

Theorem 1.3 covers the situation where \( I - K \) is one-to-one and Fredholm’s first alternative holds. In the general case where \( I - K \) is not necessarily one-to-one, the existence of solutions to

\[
u - Ku = L^{-1}_\gamma f
\]

can be determined using the identity

\[
\text{Range}(I - K) = [\text{Ker}(I - K^*)]^\perp.
\] (1.46)

Theorem 1.4 (existence of solutions to the elliptic boundary value problem - III). Under the assumptions (H), the problem (1.2) has at least one weak solution if and only if

\[
(f, v)_{L^2} = 0 \quad (1.47)
\]

for every weak solution \( v \in H^1_0(\Omega) \) of the adjoint problem (1.45).

Proof. The boundary value problem (1.2) has a weak solution provided that \( L^{-1}_\gamma f \in \text{Range}(I - K) \). By (1.46), this holds if and only if \( L^{-1}_\gamma f \) is orthogonal to every \( v \in \text{Ker}(I - K^*) \), i.e. to every solution \( v \) of the adjoint problem (1.45).

We claim that this holds if and only if \( f \) itself is orthogonal to every solution \( v \) of (1.45). Indeed, if \( v - K^*v = 0 \), one has

\[
(f, v)_{L^2} = (f, K^*v)_{L^2} = (Kf, v)_{L^2} = \gamma (L^{-1}_\gamma f, v)_{L^2}.
\]

Since \( \gamma > 0 \), this proves our claim. \( \square \)
2 Parabolic equations

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( L \) be the operator in (1.1). In addition to the standard hypotheses (H) stated at the beginning of the chapter, we now assume that the coefficients \( a^{ij} \) satisfy somewhat stronger regularity condition

\[
a^{ij} \in W^{1,\infty}(\Omega). \tag{2.1}
\]

In this section we study the parabolic initial-boundary value problem

\[
\begin{aligned}
    u_t + Lu &= 0 & \quad t > 0, \ x \in \Omega, \\
    u(t,x) &= 0 & \quad t > 0, \ x \in \partial \Omega, \\
    u(0,x) &= g(x) & \quad x \in \Omega.
\end{aligned} \tag{2.2}
\]

It is convenient to reformulate (2.2) as a Cauchy problem in the Hilbert space \( X = L^2(\Omega) \), namely

\[
\frac{d}{dt} u = Au, \quad u(0) = g, \tag{2.3}
\]

for a suitable (unbounded) linear operator \( A : L^2(\Omega) \to L^2(\Omega) \). More precisely

\[
A \equiv -L, \quad \text{Dom}(A) = \left\{ u \in H^1_0(\Omega) ; \ Lu \in L^2(\Omega) \right\}. \tag{2.4}
\]

In other words, \( u \in \text{Dom}(A) \) if \( u \) is the solution to the elliptic boundary value problem (1.2), for some \( f \in L^2(\Omega) \). In this case, \( Au = -f \).

Our eventual goal is to construct solutions of (2.3) using semigroup theory. We first consider the case where the operator \( L \) is strictly positive definite on \( H^1_0(\Omega) \). More precisely, we assume that there exists \( \beta > 0 \) such that the bilinear form \( B : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R} \) defined at (1.11) is strictly positive definite: there exists \( \beta > 0 \) such that

\[
B[u,u] \geq \beta \|u\|^2_{H^1} \quad \text{for all } u \in H^1_0(\Omega). \tag{2.5}
\]

Notice that this is certainly true if \( b^i \equiv 0 \) and \( c \geq 0 \).

**Theorem 2.1 (Semigroup of solutions of a parabolic equation - I).** Let the standard assumptions (H) hold, together with (2.1). Moreover, assume that the corresponding bilinear form \( B \) in (1.11) is strictly positive definite, so that (2.5) holds.

Then the operator \( A = -L \) generates a contractive semigroup \( \{ S_t ; \ t \geq 0 \} \) of linear operators on \( L^2(\Omega) \).

**Proof.** To prove that \( A \) generates a contraction semigroup on \( X = L^2(\Omega) \) we need to check:

(i) \( \text{Dom}(A) \) is dense in \( L^2(\Omega) \).

(ii) The graph of \( A \) is closed.

(iii) Every real number \( \lambda > 0 \) is in the resolvent set of \( A \), and \( \| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda} \).
1. To prove (i), we observe that, if \( \varphi \in C^2_c(\Omega) \), then the regularity assumptions (2.1) imply \( f = L\varphi \in L^2(\Omega) \). This proves that \( \text{Dom}(A) \) contains the subspace \( C^2_c(\Omega) \) and is thus dense in \( L^2(\Omega) \).

2. If (2.5) holds, then, by the Lax-Milgram theorem, for every \( f \in L^2(\Omega) \) there exists a unique \( u \in H^1_0(\Omega) \) such that
   \[
   B[u, v] = (f, v)_{L^2} \quad \text{for all } v \in H^1_0(\Omega).
   \]
   The map \( f \mapsto u = L^{-1}f \) is a bounded linear operator from \( L^2(\Omega) \) into \( L^2(\Omega) \).

   We now observe that the pair of functions \( (u, f) \) is in the graph of \( A \) if and only if \( (-f, u) \) lies in the graph \( L^{-1} \). Since \( L^{-1} \) is a continuous operator, its graph is closed. Hence the graph of \( A \) is closed as well.

3. According to the definition of \( A \), to prove (iii) we need to show that, for every \( \lambda > 0 \), the operator \( \lambda I - A \) has a bounded inverse with operator norm \( \| (\lambda I - A)^{-1} \| \leq 1/\lambda \). Equivalently, for every \( f \in L^2(\Omega) \), we need to show that the problem
   \[
   \begin{cases}
   \lambda u + Lu = f & x \in \Omega, \\
   u = 0 & x \in \partial \Omega,
   \end{cases}
   \]  
   has a weak solution satisfying
   \[
   \| u \|_{L^2} \leq \frac{1}{\lambda} \| f \|_{L^2}. 
   \]
   By the Lax-Milgram theorem, there exists a unique \( u \in H^1_0(\Omega) \) such that
   \[
   (\lambda u, v)_{L^2} + B[u, v] = (f, v)_{L^2} \quad \text{for all } v \in H^1_0(\Omega).
   \]
   Taking \( v = u \) in (2.8) we obtain
   \[
   \lambda \| u \|_{L^2}^2 + B[u, u] = (f, u)_{L^2} \leq \| f \|_{L^2} \cdot \| u \|_{L^2}.
   \]
   Since we are assuming \( B[u, u] \geq 0 \), this yields
   \[
   \lambda \| u \|_{L^2} \leq \| f \|_{L^2},
   \]
   proving (2.7).

   By the generation theorem, the linear operator \( A \) generates a contractive semigroup.

2.1 Representation of solutions in terms of eigenfunctions

Consider the special case where \( a^{ij} = a^{ji} \) and \( L \) is the operator in (1.20), containing only second order terms. In this case, by Poincare’s inequality, the bilinear form \( B \) in (1.22) is strictly positive definite and Theorem 2.1 can be applied.

Relying on Theorem 1.2, we can provide a representation of the semigroup trajectories, in terms of the orthonormal basis \( \{ \phi_k : k \geq 1 \} \) consisting of eigenfunctions of the compact self-adjoint operator \( L^{-1} \). By construction, for every \( k \geq 1 \) one has
   \[
   L^{-1}\phi_k = \lambda_k \phi_k,
   \]
where $\lambda_k > 0$ is the corresponding eigenvalue. Therefore

$$\phi_k \in \text{Dom}(L), \quad L\phi_k = \mu_k \phi_k \quad \mu_k = \frac{1}{\lambda_k}. \quad (2.9)$$

Notice that $\lambda_k \to 0$ and $\mu_k \to +\infty$, as $k \to \infty$. For every $k \geq 1$, the function

$$u(t) = e^{-\mu_k t} \phi_k$$

provides a $C^1$ solution to the Cauchy problem

$$\frac{d}{dt} u(t) = -Lu(t), \quad u(0) = \phi_k.$$ 

Hence, by the uniqueness of semigroup trajectories, one must have

$$S_t \phi_k = e^{-\mu_k t} \phi_k.$$ 

By linearity, for any coefficients $c_1, \ldots, c_N \in \mathbb{R}$ one has

$$S_t \left( \sum_{k=1}^{N} c_k \phi_k \right) = \sum_{k=1}^{N} c_k e^{-\mu_k t} \phi_k.$$ 

Since $S_t$ is a bounded linear operator, decomposing an arbitrary function $g \in L^2(\Omega)$ along the orthonormal basis $\{\phi_k; \ k \geq 1\}$, we thus obtain

$$S_t g = \sum_{k=1}^{\infty} e^{-\mu_k t} (g, \phi_k)_{L^2} \phi_k. \quad (2.10)$$

The above representation of semigroup trajectories is valid for every $g \in L^2(\Omega)$ and every $t \geq 0$.

**Lemma 2.1** Let $L = L_0$ be the operator at (1.20). Then for every $g \in L^2(\Omega)$ the formula (2.10) defines a map $t \mapsto u_t = S_t g$ from $[0, \infty[$ into $L^2(\Omega)$. This map is continuous for $t \in [0, \infty[$ and continuously differentiable for $t > 0$. Moreover, $u(t) \in \text{Dom}(L) \subseteq H^1_0(\Omega)$ for every $t > 0$ and there holds

$$\frac{d}{dt} u(t) = Lu(t) \quad \text{for all } t > 0. \quad (2.11)$$

**Proof.** Let $g \in L^2(\Omega)$. Since $\mu_k > 0$ for every $k$, it is clear that

$$\left| e^{-\mu_k t} (g, \phi_k)_{L^2} \right|^2 \leq (g, \phi_k)_{L^2}^2.$$ 

Therefore,

$$\sum_{k \geq 1} \left| e^{-\mu_k t} (g, \phi_k)_{L^2} \right|^2 \leq \sum_{k \geq 1} (g, \phi_k)_{L^2}^2 = \|g\|_{L^2}^2 < \infty.$$ 

Hence the series in (2.10) is convergent, uniformly for $t \geq 0$. In particular, since the partial sums are continuous functions of time, the map $t \mapsto S_t g$ is continuous as well.
2. We claim that, even if \( g \notin H^1_0(\Omega) \), one always has
\[
S_tg \in \text{Dom}(L) \subseteq H^1_0(\Omega) \quad \text{for all } t > 0.
\]
(2.12)
Indeed, a function \( u = \sum_k c_k \phi_k \) lies in \( \text{Dom}(L) \) if and only if the coefficients \( c_k \) satisfy
\[
\sum_k (c_k \mu_k)^2 < \infty.
\]
In the case where \( c_k(t) \equiv e^{-\mu_k t}(g, \phi_k)_{L^2} \), we have the estimate
\[
\sum_k (\mu_k c_k(t))^2 \leq \sup_k (\mu_k e^{-\mu_k t})^2 \cdot \sum_k (g, \phi_k)_{L^2}^2.
\]
(2.13)
An elementary calculation now shows that, for \( \xi \geq 0 \) and \( t \) fixed, the function \( \xi \mapsto \xi e^{-t \xi} \) attains its global maximum at \( \xi = 1/t \). In particular,
\[
\mu_k e^{-\mu_k t} \leq \max_{\xi \geq 0} \xi e^{-t \xi} = \frac{1}{et}.
\]
Using this bound in (2.13) we obtain
\[
\sum_{k=1}^{\infty} (\mu_k c_k(t))^2 \leq \frac{1}{e^2 t^2} \|g\|_{L^2}^2.
\]
Hence the series defining \( Lu(t) \) is convergent. This implies \( u(t) \in \text{Dom}(L) \), for each \( t > 0 \).

3. Differentiating the series (2.10) term by term, and observing that the series of derivatives is also convergent, we achieve (2.11). \( \square \)

**Example 8.3.** As in Example 8.1, let \( \Omega = [0, \pi] \subseteq \mathbb{R} \) and \( L_0 u = -u_{xx} \). Given \( g \in L^2([0, \pi]) \), consider the parabolic initial-boundary value problem
\[
\begin{cases}
    u_t = u_{xx} & t > 0, \ 0 < x < \pi, \\
    u(0, x) = g(x) & 0 < x < \pi, \\
    u(t, 0) = u(t, \pi) = 0.
\end{cases}
\]
In this special case, the formula (2.10) yields the solution as the sum of a Fourier sine series:
\[
u(t, x) = \sum_{k=1}^{\infty} \frac{2}{\pi} e^{-k^2 t} \left( \int_0^\pi g(y) \sin ky \, dy \right) \sin kx.
\]

2.2 More general operators
To motivate the following construction, we begin with a finite dimensional example. Let \( L \) be an \( n \times n \) matrix and consider the linear ODE on \( \mathbb{R}^n \)
\[
\frac{d}{dt} x(t) = -Lx(t).
\]
(2.14)
If $L$ is positive definite, i.e. if $\langle Lx, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$, then $-L$ generates a contractive semigroup. Indeed
\[
\frac{d}{dt} |x(t)|^2 = 2 \left\langle \frac{d}{dt} x(t), x(t) \right\rangle = 2 \langle -Lx(t), x(t) \rangle \leq 0,
\]
showing that the Euclidean norm of a solution does not increase in time.

Next, let $L$ be an arbitrary matrix. We can then find a number $\gamma \geq 0$ large enough so that the matrix $L + \gamma I$ is positive definite (and hence it generates a contractive semigroup). In this case, if $x(t) = e^{-tL}x(0)$ is a solution to (2.14), writing $-L = \gamma I - (L + \gamma I)$ one obtains
\[
|x(t)| = |e^{-Lt}x(0)| = |e^{\gamma t - (L+\gamma I)t}x(0)| = e^{\gamma t} |e^{-(L+\gamma I)t}x(0)| \leq e^{\gamma t} |x(0)|.
\] (2.15)
According to (2.15), the operator $-L$ generates a semigroup of type $\gamma$.

We shall work out a similar construction in the case where $L$ is a general elliptic operator, as in (1.1), and the corresponding bilinear form $B[u,v]$ in (1.11) is not necessarily positive definite. According to Lemma 1.2, there exists a constant $\gamma > 0$ large enough so that the bilinear form
\[
B_{\gamma}[u,v] = B[u,v] + \gamma(u,v)_{L^2}
\] (2.16)
is strictly positive definite on $H^1_0(\Omega)$. We can thus define
\[
L_\gamma u = Lu + \gamma u, \quad B_{\gamma}[u,v] = B[u,v] + \gamma(u,v)_{L^2}.
\]
The parabolic equation in (2.2) can now be written as
\[
u_t = -L_\gamma u + \gamma u.
\]
By the previous analysis, the operator $A_\gamma \doteq -(L + \gamma I)$ generates a contractive semigroup of linear operators, say $\{S_t^{(\gamma)}; \ t \geq 0\}$. Therefore, the operator $A \doteq -L = \gamma I - L_\gamma$ defined as in (2.4) generates a semigroup of type $\gamma$. Namely $\{S_t; \ t \geq 0\}$, with
\[
S_t = e^{\gamma t} S_t^{(\gamma)} \quad t \geq 0.
\]
Summarizing the above analysis, we have

**Theorem 8.4 (semigroup of solutions of a parabolic equation - II).** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that the operator $L$ in (1.1) satisfies the regularity conditions (2.1) and the uniform ellipticity condition (1.4).

Then the operator $A$ defined at (2.4) generates a semigroup $\{S_t; \ t \geq 0\}$ of linear operators on $L^2(\Omega)$.

Having constructed a semigroup $\{S_t; \ t \geq 0\}$ generated by the operator $A$, one needs to understand in which sense a trajectory of the semigroup $t \mapsto u(t) = S_t f$ provides a solution to the parabolic equation (2.2). In the case where $L = L_0$ is the symmetric operator defined at (1.20), the representation (2.10) provides all the needed information. Indeed, according to
Lemma 8.3, for every initial data \( g \in L^2(\Omega) \) the solution \( t \mapsto u(t) = S_t g \) is a \( C^1 \) map, which takes values in \( \text{Dom}(L_0) \) and satisfies (2.11) for every \( t > 0 \).

A similar result can be proved for general elliptic operators of the form (1.1). However, this analysis is beyond the scope of the present notes. Here we shall only make a few remarks:

(1) - Initial condition. The map \( t \mapsto u(t) = S_t g \) is continuous from \([0, \infty[\) into \( L^2(\Omega) \) and satisfies \( u(0) = g \). The initial condition in (2.2) is thus satisfied as an identity between functions in \( L^2(\Omega) \).

(2) - Regular solutions. If \( g \in \text{Dom}(A) \), then \( u(t) = S_t g \in \text{Dom}(A) \) for all \( t \geq 0 \). Moreover, the map \( t \mapsto u(t) \) is continuously differentiable and satisfies the O.D.E. (2.3) at every time \( t > 0 \). Since \( \text{Dom}(A) \subset H^1_0(\Omega) \), this also implies that \( u(t) \) satisfies the correct boundary conditions, for all \( t \geq 0 \).

(3) - Distributional solutions. Given a general initial condition \( f \in L^2(\Omega) \), one can construct a sequence of initial data \( f_m \in \text{Dom}(A) \) such that \( \|f_m - f\|_{L^2} \to 0 \) as \( m \to \infty \). In this case, if the semigroup is of type \( \gamma \), we have

\[
\|S_t f_m - S_t f\|_{L^2} \leq e^{\gamma t} \|f_m - f\|_{L^2}.
\]

Therefore the trajectory \( t \mapsto u(t) = S_t f \) is the limit of a sequence of \( C^1 \) solutions \( t \mapsto u_m(t) = S_t f_m \). The convergence is uniform for \( t \) in bounded sets.

Relying on these approximations, we now show that the function \( u = u(t, x) \) provides a solution to the parabolic equation

\[
\frac{du}{dt} = \sum_{i,j=1}^n (a^{ij}(x)u_x)_x - \sum_{i=1}^n b^i(x)u_x - c(x)u
\]

in distributional sense. Namely, for every test function \( \varphi \in C_c^\infty(\Omega \times ]0, \infty[) \), one has

\[
\int \int \left\{ u \varphi_t + \sum_{i,j=1}^n u(a^{ij}(x)\varphi_x)_x + \sum_{i=1}^n u(b^i(x)\varphi_x) - c u \varphi \right\} dx dt = 0. \tag{2.18}
\]

To prove (2.18), consider a sequence of initial data \( f_m \in \text{Dom}(A) \) such that \( \|f_m - f\|_{L^2} \to 0 \). Then, for any fixed time interval \([0, T]\), the trajectories \( t \mapsto u_m(t) = S_t f_m \) converge to the continuous trajectory \( t \mapsto u(t) = S_t f \) in \( C^0([0, T]; L^2(\Omega)) \). Since each \( u_m \) is clearly a solution in distributional sense, writing

\[
\int \int \left\{ u_m \varphi_t + \sum_{i,j=1}^n u_m(a^{ij}(x)\varphi_x)_x + \sum_{i=1}^n u_m(b^i(x)\varphi_x) - c u_m \varphi \right\} dx dt = 0,
\]

and letting \( m \to \infty \) we obtain (2.18).
3 Hyperbolic equations

Consider a linear hyperbolic initial-boundary value problem, of the form

\[
\begin{aligned}
  u_{tt} + L_0 u &= 0 & t \in \mathbb{R}, & x \in \Omega, \\
  u(t, x) &= 0 & t \in \mathbb{R}, & x \in \partial \Omega, \\
  u(0, x) = f(x), & \quad u_t(0, x) = g(x) & x \in \Omega,
\end{aligned}
\]  

(3.19)

where \( L_0 \) is the second order elliptic operator

\[
L_0 u = -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j}.
\]  

(3.20)

We assume that the coefficients \( a_{ij} \) satisfy

\[
a_{ij} = a_{ji} \in W^{1,\infty}(\Omega), \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n.
\]  

(3.21)

According to Lemma 8.2, the space \( L^2(\Omega) \) admits an orthonormal basis \( \{\phi_k; \ k \geq 1\} \) consisting of eigenfunctions of \( L_0 \), so that

\[
\phi_k \in \text{Dom}(L_0), \quad L_0 \phi_k = \mu_k \phi_k,
\]  

(3.22)

for a sequence of eigenvalues \( \mu_k \to +\infty \) as \( k \to \infty \).

It is convenient to reformulate (3.19) as a first order system, setting \( v = u_t \). On the product space

\[
X = H^1_0(\Omega) \times L^2(\Omega),
\]  

(3.23)

we thus consider the evolution problem

\[
\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} f \\ g \end{pmatrix}.
\]  

(3.24)

A semigroup of solutions of (3.24) will be constructed using the eigenfunctions \( \phi_k \). In the special case where

\[
f = a_k \phi_k, \quad g = b_k \phi_k,
\]  

(3.25)

for a given \( k \geq 1 \) and \( a_k, b_k \in \mathbb{R} \), an explicit solution of (3.24) is found in the form

\[
u(t) = a(t) \phi_k, \quad v(t) = u_t(t) = a'(t) \phi_k,
\]  

where the coefficient \( a(t) \) satisfies

\[
a''(t) + \mu_k a(t) = 0, \quad a(0) = a_k, \quad a'(0) = b_k.
\]  

An elementary computation yields

\[
a(t) = a_k \cos(\sqrt{\mu_k} t) + \frac{b_k}{\sqrt{\mu_k}} \sin(\sqrt{\mu_k} t).
\]  

Hence

\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\mu_k} t) & \frac{1}{\sqrt{\mu_k}} \sin(\sqrt{\mu_k} t) \\ -\sqrt{\mu_k} \sin(\sqrt{\mu_k} t) & \cos(\sqrt{\mu_k} t) \end{pmatrix} \begin{pmatrix} a_k \phi_k \\ b_k \phi_k \end{pmatrix}.
\]  

(3.26)
Observe that $t \mapsto (u(t), v(t))$ is a continuously differentiable map from $\mathbb{R}$ into $H^0_0(\Omega) \times L^2(\Omega)$ which satisfies the initial conditions and the differential equation in (3.24).

By linearity, more general solutions can be obtained by taking linear combinations of solutions of the form (3.26). This motivates the definition

$$S_t \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix} \cos(\sqrt{\mu_k} t) \\ -\sqrt{\mu_k} \sin(\sqrt{\mu_k} t) \end{pmatrix} \begin{pmatrix} (f, \phi_k)_{L^2(\Omega)} \phi_k \\ (g, \phi_k)_{L^2(\Omega)} \phi_k \end{pmatrix}.$$  

(3.27)

**Theorem 3.1 (solutions of a linear hyperbolic problem).** In the above setting, the formula (3.27) defines a strongly continuous group of bounded linear operators $\{S_t; \ t \in \mathbb{R}\}$ on the space $X = H^0_0(\Omega) \times L^2(\Omega)$. Each operator $S_t : X \mapsto X$ is an isometry w.r.t. the equivalent norm

$$\|(u, v)\|_X = \left( B_0[u, u] + \|v\|_{L^2}^2 \right)^{1/2}.  \quad (3.28)$$

**Remark 3.1** Consider an elastic membrane which occupies a region $\Omega$ in the plane. Let $u(t, x)$ denote the vertical displacement of a point on this membrane, at time $t$. Then the square of the norm $\|(u, u_t)\|_X^2$ can be interpreted as the total energy of the vibrating membrane. Indeed, the term $B_0[u, u]$ describes an elastic potential energy, while $\|u_t\|_{L^2}^2$ yields the kinetic energy.

**Proof of Theorem 3.1.**

1. The equivalence between the norm (3.28) and the standard product norm

$$\|(u, v)\|_{H^1 \times L^2} = \left( \|u\|_{H^1}^2 + \|v\|_{L^2}^2 \right)^{1/2}$$

is an immediate consequence of (1.23).

2. Let the functions $f, g$ be given by

$$f = \sum_{k=1}^{\infty} a_k \phi_k, \quad g = \sum_{k=1}^{\infty} b_k \phi_k \quad \quad (a_k = (f, \phi_k)_{L^2(\Omega)}, \ b_k = (g, \phi_k)_{L^2(\Omega)}).$$

Then

$$B_0[f, f] = \sum_{k=1}^{\infty} (\mu_k a_k \phi_k, a_k \phi_k)_{L^2} = \sum_{k=1}^{\infty} \mu_k a_k^2, \quad \quad \|g\|_{L^2}^2 = \sum_{k=1}^{\infty} b_k^2.$$

Therefore $\begin{pmatrix} f \\ g \end{pmatrix} \in X$ if and only if

$$\sum_{k=1}^{\infty} \mu_k a_k^2 < \infty, \quad \sum_{k=1}^{\infty} b_k^2 < \infty. \quad (3.29)$$
At any time \( t \in \mathbb{R} \), we have
\[
\left\| S_t \left( \begin{array}{c} f \\ g \end{array} \right) \right\|_X^2 = \sum_{k=1}^{\infty} \mu_k a_k^2(t) + \sum_{k=1}^{\infty} b_k(t)^2 ,
\] (3.30)
where, according to (3.27),
\[
\begin{cases}
  a_k(t) = \cos(\sqrt{\mu_k} t) a_k + \frac{1}{\sqrt{\mu_k}} \sin(\sqrt{\mu_k} t) b_k , \\
  b_k(t) = a_k'(t) = -\sqrt{\mu_k} \sin(\sqrt{\mu_k} t) a_k + \cos(\sqrt{\mu_k} t) b_k .
\end{cases}
\] (3.31)

If \( \left( \begin{array}{c} f \\ g \end{array} \right) \in X \), then for every \( t \in \mathbb{R} \) the series defining \( S_t \left( \begin{array}{c} f \\ g \end{array} \right) \) in (3.27) is convergent. Using (3.31) inside (3.30), we obtain
\[
\left\| S_t \left( \begin{array}{c} f \\ g \end{array} \right) \right\|_X^2 = \sum_{k=1}^{\infty} \mu_k a_k^2(t) + \sum_{k=1}^{\infty} b_k(t)^2 = \left\| \left( \begin{array}{c} f \\ g \end{array} \right) \right\|_X^2 .
\] (3.32)
This shows that each linear operator \( S_t \) is an isometry w.r.t. the equivalent norm \( \| \cdot \|_X \).

3. It is easy to check that the family of linear operators \( \{ S_t ; \ t \in \mathbb{R} \} \) satisfies the group properties
\[
S_0 \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{c} f \\ g \end{array} \right) , \quad S_t S_s \left( \begin{array}{c} f \\ g \end{array} \right) = S_{t+s} \left( \begin{array}{c} f \\ g \end{array} \right) \quad t, s \in \mathbb{R} .
\]

To complete the proof, we need to show that, for \( f, g \) fixed, the map \( t \mapsto S_t \left( \begin{array}{c} f \\ g \end{array} \right) \) is continuous from \( \mathbb{R} \) into \( X \). But this is clear, because the above map is the uniform limit of the continuous maps
\[
t \mapsto S_t \left( \begin{array}{c} f_m \\ g_m \end{array} \right) , \quad f_m = \sum_{k=1}^{m} (f, \phi_k)_{L^2} \phi_k , \quad g_m = \sum_{k=1}^{m} (g, \phi_k)_{L^2} \phi_k ,
\] (3.33)
as \( m \to \infty \).

Having constructed the group of operators \( \{ S_t ; \ t \in \mathbb{R} \} \), one needs to understand in which sense the trajectories
\[
t \mapsto \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right) = S_t \left( \begin{array}{c} f \\ g \end{array} \right)
\] (3.34)
provide a solution to the hyperbolic initial-boundary value problem (3.19).

(1) - Initial and boundary conditions. Consider an arbitrary initial data \( \left( \begin{array}{c} f \\ g \end{array} \right) \in X = H^1_0(\Omega) \times L^2(\Omega) \). By the continuity of the map (3.34), it follows that
\[
\| u(t) - f \|_{H^1} \to 0 , \quad \| v(t) - g \|_{L^2} \to 0 \quad \text{as} \quad t \to 0 .
\]
Hence the initial conditions in (3.19) are satisfied.

Moreover, by the definition of the space $X$, we have $u(t) \in H^1_0(\Omega)$ for all $t \geq 0$. Hence the boundary conditions in (3.19) are also satisfied.

(2) - Regular solutions. If both functions $f$ and $g$ are finite linear combinations of the eigenfunctions $\phi_k$, then the corresponding trajectory (3.34) is a continuously differentiable map from $[0, \infty[$ into $X$, and it satisfies the ODE (3.24) at all times $t > 0$.

(3) - Distributional solutions. Given general initial data $f \in H^1_0(\Omega)$ and $g \in L^2(\Omega)$, one can construct a sequence of approximations $f_m, g_m$ as in (3.33), so that $\|f - f_m\|_{H^1} \to 0$, $\|g - g_m\|_{L^2} \to 0$ as $m \to \infty$. The corresponding semigroup trajectories $t \mapsto \begin{pmatrix} u_m(t) \\ v_m(t) \end{pmatrix} = S_t \begin{pmatrix} f_m \\ g_m \end{pmatrix}$ converge to the trajectory (3.34), uniformly for $t \in \mathbb{R}$.

Relying on these approximations, we now show that the function $u = u(t, x)$ provides a solution to the parabolic equation

$$u_{tt} = \sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j}$$

in distributional sense. Indeed, consider any test function $\varphi \in C^\infty_c(]0, \infty[ \times \Omega)$. Since each function $u_m = u_m(t, x)$ is a distributional solution of (3.19), there holds

$$\iint \left\{ u_m \varphi_{tt} + \sum_{i,j=1}^{n} a^{ij}(x) (u_m)_{x_i} \varphi_{x_j} \right\} dx dt = 0.$$

Letting $m \to \infty$ and using the convergence $\|u_m(t) - u(t)\|_{H^1} \to 0$, uniformly for $t \in \mathbb{R}$, we obtain

$$\iint \left\{ u \varphi_{tt} + \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} \varphi_{x_j} \right\} dx = 0.$$

Example 8.4. As in Example 8.1, let $\Omega \equiv ]0, \pi[ \subset \mathbb{R}$ and $L_0 u = -u_{xx}$. Given $f \in H^1_0(\Omega)$ and $g \in L^2(]0, \pi[)$, consider the hyperbolic initial-boundary value problem

$$\begin{cases}
    u_{tt} = u_{xx} & t > 0, \ 0 < x < \pi, \\
    u(0, x) = f(x), \ u_t(0, x) = g(x) & 0 < x < \pi, \\
    u(t, 0) = u(t, \pi) = 0.
\end{cases}$$

In this special case, the formula (3.27) yields the solution as the sum of a Fourier sine series:

$$u(t, x) = \sum_{k=1}^{\infty} \frac{2}{\pi} \left[ \cos kt \left( \int_0^\pi f(y) \sin ky \, dy \right) + \frac{\sin kt}{k} \left( \int_0^\pi g(y) \sin ky \, dy \right) \right] \sin kx.$$
4 Problems

1. Let \( \Omega = \{(x, y) : x^2 + y^2 < 1 \} \) be the open unit disc in \( \mathbb{R}^2 \). Prove that, for every bounded measurable function \( f = f(x, y) \), the problem

\[
\begin{align*}
\begin{cases}
u_{xx} + xu_{xy} + u_{yy} &= f & \text{on } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{cases}
\]

has a unique weak solution.

2. Let \( \Omega = \{(x, y) : x^2 + y^2 < 1 \} \) be the unit disc in \( \mathbb{R}^2 \). On the space \( X = H^1_0(\Omega) \), consider the inner product

\[
\langle u, v \rangle \doteq \int_{\Omega} \left[ u_x v_x + 2u_y v_y + y(u_x v_y + u_y v_x) \right] dx.
\]

(i) Prove that \( \langle \cdot, \cdot \rangle \) is indeed an inner product on \( X \), which makes \( X \) into a Hilbert space.

(ii) Given \( f \in L^2(\Omega) \), show that there exists a unique \( u \in X \) such that

\[
\langle u, v \rangle = \int_{\Omega} f v \, dx \quad \text{for all } v \in X = H^1_0(\Omega).
\]

What elliptic equation does \( u \) solve?

3. On a bounded open set \( \Omega \subset \mathbb{R}^n \) consider an elliptic operator of the form

\[
Lu \doteq - \sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + c(x)u,
\]

assuming that the coefficients \( a^{ij}, c \) satisfy (1.3) and (1.4).

(i) Prove that the space \( L^2(\Omega) \) admits an orthonormal basis \( \{\phi_k; \ k \geq 1\} \) consisting of eigenfunctions of \( L \), so that

\[
\phi_k \in H^1_0(\Omega), \quad L\phi_k = \mu_k \phi_k
\]

for all \( k \geq 1 \). The corresponding eigenvalues satisfy \( \mu_k \to +\infty \) as \( k \to \infty \).

(ii) Show that the general solution to the parabolic initial-boundary value problem (2.2) can be written as

\[
S_{t}g = \sum_{k=1}^{\infty} e^{-\mu_k t} (g, \phi_k)_{L^2} \phi_k.
\]

4. Prove that a representation similar to (3.27) continues to hold, for the solution to the more general hyperbolic initial-boundary value problem

\[
\begin{align*}
\begin{cases}
u_{tt} + Lu &= 0 & \text{in } \mathbb{R}, \ x \in \Omega, \\
u(t, x) &= 0 & \text{in } \mathbb{R}, \ x \in \partial \Omega, \\
u(0, x) &= f(x), \quad u_t(0, x) = g(x) & x \in \Omega,
\end{cases}
\]

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where $L$ is the elliptic operator in (4.35).

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $\beta > 0$ be the constant in Poincaré’s inequality:
\[
\|u\|_{L^2(\Omega)} \leq \beta \|\nabla u\|_{L^2(\Omega)}.
\]
(i) Establish a lower bound on the eigenvalues of the operator $Lu = -\Delta u$. More precisely, if $\phi \in H^1_0(\Omega)$ provides a weak solution to
\[
\begin{cases}
-\Delta \phi = \mu \phi & x \in \Omega, \\
\phi = 0 & x \in \partial \Omega,
\end{cases}
\]
prove that $\mu \geq 1/\beta^2$.

(ii) Prove that the solution of the parabolic initial-boundary value problem
\[
\begin{cases}
\frac{du}{dt} = \Delta u & t > 0, \ x \in \Omega, \\
u(t, x) = 0 & t > 0, \ x \in \partial \Omega, \\
u(0, x) = g(x) & x \in \Omega,
\end{cases}
\]
decays to zero as $t \to \infty$. Indeed, $\|u(t)\|_{L^2} \leq e^{-t/\beta^2}\|g\|_{L^2}$ for every $t \geq 0$.

6. On the interval $[0, T]$, consider the Sturm-Liouville eigenvalue problem
\[
\begin{cases}
(p(t)u')' + q(t)u = \mu u & 0 < t < T, \\
u(0) = u(T) = 0.
\end{cases}
\]
Assume that
\[
p \in C^1([0, T]), \quad q \in C^0([0, T]), \quad p(t) \geq \theta > 0 \quad \text{for all} \ t.
\]
Prove that the space $L^2([0, T])$ admits an orthonormal basis $\{\phi_k; \ k \geq 1\}$ where each $\phi_k$ satisfies (4.36), for a suitable eigenvalue $\mu_k$. Moreover, $\mu_k \to -\infty$ as $k \to \infty$.

7. In Lemma 8.2, take $L_0u = -\Delta u$. Let $\{\phi_k; \ k \geq 1\}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $L_0^{-1}$. Show that in this special case the eigenfunctions $\phi_k$ are mutually orthogonal also w.r.t. the inner product in $H^1$, namely $(\phi_j, \phi_k)_{H^1} = 0$ whenever $j \neq k$.

8. On a bounded open set $\Omega \subset \mathbb{R}^n$, let $0 < \mu_1 \leq \mu_2 \leq \cdots$ be the eigenvalues of the operator $Lu = -\Delta u$ (with zero boundary conditions).
(i) Given any $\varphi \in C_0^\infty(\Omega)$, prove that
\[
\int_\Omega |\nabla \varphi|^2 \, dx \geq \mu_1 \int_\Omega |\varphi|^2.
\]

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(ii) Prove that the best possible constant in Poincare’s inequality
\[ \|u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \quad u \in H^1_0(\Omega) \] (4.37)
is \( C = 1/\mu_1 \).

9. Let \( \Omega \subseteq \tilde{\Omega} \subseteq \mathbb{R}^n \) be bounded open sets. Let \( 0 < \mu_1 \leq \mu_2 \leq \cdots \) be the eigenvalues of the operator \(-\Delta\) on the domain \( \Omega \), and let \( 0 < \tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots \) be the eigenvalues of the operator \(-\Delta\) on \( \tilde{\Omega} \). Prove that \( \tilde{\mu}_1 \leq \mu_1 \).

10. Consider the open rectangle \( Q \equiv \{(x, y); \ 0 < x < a, \ 0 < y < b\} \). Define the functions
\[ \phi_{m,n}(x, y) = \sqrt{\frac{2}{ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad m, n \geq 1. \]

(i) Check that \( \phi_{m,n} \in H^1_0(Q) \). Moreover, prove that the countable set of functions \( \{\phi_{m,n}; \ m, n \geq 1\} \) is an orthonormal basis of \( L^2(Q) \) consisting of eigenfunctions of the elliptic operator \( Lu = -\Delta u \). Compute the corresponding eigenvalues \( \mu_{m,n} \).

(ii) If \( \Omega \subset \mathbb{R}^2 \) is an open domain contained inside a rectangle \( Q \) with sides \( a, b \), prove that
\[ \|u\|_{L^2(\Omega)} \leq \frac{ab}{\pi \sqrt{a^2 + b^2}} \|\nabla u\|_{L^2(\Omega)} \quad \text{for all} \ u \in H^1_0(\Omega). \]

11. (Galerkin approximations) Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Given \( f \in L^2(\Omega) \), consider the elliptic problem
\[ \begin{cases} -\Delta \phi = f & x \in \Omega, \\ \phi = 0 & x \in \partial \Omega, \end{cases} \]
Let \( \{\varphi_1, \ldots, \varphi_m\} \) be any set of linearly independent functions in \( H^1_0(\Omega) \). Construct an approximate solution by setting
\[ U = \sum_{k=1}^{m} c_k \varphi_k \]
where the coefficients \( c_1, \ldots, c_m \) are chosen so that
\[ B[U, \varphi_j] = \int_{\Omega} \nabla U \cdot \nabla \varphi_j \, dx = (f, \varphi_j)_{L^2} \quad j = 1, \ldots, m. \] (4.38)
Show that (4.38) can be written as an algebraic system of \( m \) linear equations for the \( m \) variables \( c_1, \ldots, c_m \). Prove that this system has a unique solution.

12. Let \( \Omega = \{(x, y); \ x^2 + y^2 < 1\} \) be the open unit disc in \( \mathbb{R}^2 \), and let \( u \) be a smooth solution to the equation
\[ u_{tt} = u_{xx} + x u_{xy} + 3u_{yy} \quad \text{on} \ \Omega \times [0, T], \] (4.39)
\[ u = 0 \quad \text{on } \partial \Omega \times [0, T]. \]

(i) Write the equation (4.39) in the form \( u_{tt} + Lu = 0 \), proving that the operator \( L \) is uniformly elliptic on the domain \( \Omega \).

(ii) Define a suitable energy \( e(t) = [\text{kinetic energy}] + [\text{elastic potential energy}] \), and check that it is constant in time.

13. Consider the differential operator on \( \mathbb{R}^2 \)

\[ Lu = -(xu_x)_x - (yu_y)_y + 2u_{xy} + 3(u_x + u_y) - 6u. \]

(i) At which points \((x, y)\) is \( L \) elliptic?

(ii) Determine for which open bounded subsets \( \Omega \subset \mathbb{R}^2 \) one can say that operator \( L \) is uniformly elliptic on \( \Omega \).

14. Let \( \Omega = \{(x, y) \mid x^2 + y^2 < 1\} \) be the open unit disc in \( \mathbb{R}^2 \), and let \( u \) be a smooth solution to the equation

\[ u_{tt} = u_{xx} + 2xu_{xy} + 3u_{yy} + u_y \quad \text{on } \Omega \times [0, T], \]

\[ u = 0 \quad \text{on } \partial \Omega \times [0, T]. \] (4.40)

(i) Write the equation (4.40) in the form \( u_{tt} + Lu = 0 \), proving that the operator \( L \) is uniformly elliptic on the domain \( \Omega \).

(ii) Define a suitable energy \( E(t) = [\text{kinetic energy}] + [\text{elastic potential energy}] \), and check that it is constant in time.

15. Consider the homogenous linear elliptic operator \( L_0 \) in (??) assuming that (1.4) holds, together with \( a^{ij} = a^{ji} \in L^\infty(\Omega) \). Work out the following alternative proof of Theorem 1.1.

(i) Show that the bilinear functional \( B_0 \) in (1.22) is an inner product on \( H^1_0(\Omega) \). The corresponding norm

\[ \|u\|_\diamond = \left( \sum_{i,j=1}^n a^{ij}(x)u_{x_i}u_{x_j} \right)^{1/2} \]

is equivalent to the \( H^1 \) norm. Namely

\[ \frac{1}{C} \cdot \|u\|_{H^1} \leq \|u\|_\diamond \leq C\|u\|_{H^1} \quad \text{for all} \quad u \in H^1_0. \]
Call $H_\diamond$ the Hilbert space $H^1_0$ endowed with this equivalent norm. Following the proof of Lemma 1.1, construct the solution of (1.19) as $u = i^* f$, using the following diagram:

\[
\begin{align*}
H_\diamond(\Omega) & \xrightarrow{i} L^2(\Omega) \\
H_\diamond(\Omega) &= [H_\diamond(\Omega)]^* \\& \leftrightarrow [L^2(\Omega)]^* = L^2(\Omega).
\end{align*}
\]