Hyperbolic Systems of Conservation Laws

III - Uniqueness and continuous dependence and viscous approximations

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Continuous dependence on initial data

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

Given two solutions \( u, v \), estimate the difference \( \|u(t) - v(t)\|_{L^1} \)

Standard approach: set \( w = u - v \), show that

\[ \frac{d}{dt} \|w(t)\| \leq C \|w\| \]

hence \( \|w(t)\| \leq e^{Ct} \|w(0)\| \)

Works for Lipschitz solutions, not in the presence of shocks
For two solutions $u, v$ of a hyperbolic system containing shocks, the $L^1$ distance can increase rapidly during short time intervals.
The scalar conservation law

\[ u_t + f(u)_x = 0 \]

**Contraction property** (A.I. Volpert, 1967, S. Kruzkhov, 1970)

\[ \|u(t) - v(t)\|_{L^1} \leq \|u(0) - v(0)\|_{L^1} \quad \text{for all } t \geq 0 \]
The $L^1$ distance between continuous solutions remains constant.

Wave speed

$u(0)$

$v(0)$

$x$

$u(t)$

$v(t)$

$x$
The $L^1$ distance decreases when a shock in one solution crosses the graph of the other solution.
Linear Hyperbolic Systems

\[ u_t + Au_x = 0 \quad u \in \mathbb{R}^n \]

\[ \|u\|_{L^1} = \int |u(x)| \, dx \]

Right, left eigenvectors: \( l_i A = \lambda_i l_i \quad Ar_i = \lambda_i r_i \)

Equivalent norm: \[ \|u\|_A = \sum_{i=1}^{n} \|l_i \cdot u\|_{L^1} \]

\( u, \nu \) solutions \( \implies \) \( w = u - \nu \) satisfies \( w_t + Aw_x = 0 \)

\[ \|w(t)\|_A = \|w(0)\|_A \quad \text{for all } t \in \mathbb{R} \]
Continuous dependence on the initial data

\[ u_t + f(u)_x = 0 \quad \text{with} \quad u(0, x) = \bar{u}(x) \]

A domain of integrable functions with small total variation:

\[ \mathcal{D} = \text{cl}\left\{ u \in L^1(\mathbb{R}; \mathbb{R}^n); \quad u \text{ is piecewise constant,} \quad V(u) + C_0 \cdot Q(u) < \delta_0 \right\} \]

**Theorem.** For every \( \bar{u} \in \mathcal{D}, \) front tracking approximations converge to a unique limit solution \( u : [0, \infty[ \mapsto \mathcal{D}. \)

The map \((\bar{u}, t) \mapsto u(t, \cdot) = S_t \bar{u}\) is a uniformly Lipschitz semigroup, i.e.:

\[ S_0 \bar{u} = \bar{u}, \quad S_s(S_t \bar{u}) = S_{s+t} \bar{u} \]
Proof: by a homotopy method

$2 \times 2$ systems : A. Bressan and R. M. Colombo, 1994

$n \times n$ systems : A. Bressan, G. Crasta, B. Piccoli, 1996
Alternative method: a non-linear distance functional

(A. Bressan, T.P. Liu, T. Yang, 1998)

Construct a functional $\Phi : \mathcal{D} \times \mathcal{D} \mapsto \mathbb{R}_+$ such that

- $\Phi$ is equivalent to the $L^1$ distance
  \[
  \frac{1}{C} \cdot \|v - u\|_{L^1} \leq \Phi(u, v) \leq C \cdot \|v - u\|_{L^1}
  \]
- $\Phi$ is non-increasing in time, along couples of solutions
  \[
  \Phi(u(t), v(t)) \leq \Phi(u(s), v(s)) \quad 0 \leq s \leq t
  \]
Construction of the distance functional

For each $x \in \mathbb{R}$, decompose the jump $u(x), v(x)$ as a concatenation of shocks:

$$v(x) = S_n(q_n(x)) \circ \cdots \circ S_1(q_1(x))(u(x))$$

$q_1(x), \ldots, q_n(x) =$ scalar components of the jump

Remark. In the linear case $u_t + Au_x = 0$, one has

$$q_i(x) = l_i \cdot (v(x) - u(x))$$
\[ \Phi(u, v) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |q_i(x)| W_i(x) \, dx \]

Weight functions:

\[ W_i(x) = 1 + \kappa_1 \cdot \left[ \text{total strength of waves in } u \text{ and in } v \right. \]

\[ \left. \text{which approach the } i\text{-wave } q_i(x) \right] \]

\[ + \kappa_2 \cdot \left[ \text{wave interaction potentials of } u \text{ and of } v \right] \]

\[ = 1 + \kappa_1 A_i(x) + \kappa_2 [Q(u) + Q(v)] \]
Uniqueness of Solutions

\[ u_t + f(u)_x = 0 \quad \text{and} \quad u(0, x) = \bar{u}(x) \]

**Question:** Do all entropy-admissible weak solutions coincide with the ones obtained by front tracking approximations?

**Key step:** estimate the distance between a weak solution \( t \mapsto u(t) \) and a trajectory \( t \mapsto S_t \bar{u} \) of the semigroup obtained taking limits of front tracking approximations.
Theorem. Let \( S : \mathcal{D} \times [0, \infty[ \mapsto \mathcal{D} \) be a Lipschitz semigroup satisfying
\[
\| S_t u - S_s v \| \leq L \cdot \| u - v \| + L' \cdot | t - s |
\]
Then, for every Lipschitz continuous map \( w : [0, T] \mapsto \mathcal{D} \) one has
\[
\| w(T) - S_T w(0) \| \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0^+} \frac{\| w(t + h) - S_h w(t) \|}{h} \right\} dt
\]
\[
= L \cdot \int_0^T [\text{instantaneous error rate at time } t] \, dt
\]
Instantaneous error rate for a front tracking approximation

\[
\frac{1}{h} \left\| w(\tau + h) - S_h w(\tau) \right\|_{L^1} = \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{R}} |\sigma_\alpha|^2 + \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{NP}} |\sigma_\alpha|
\]
Question: among all weak solutions of

$$u_t + f(u)_x = 0$$

which ones are obtained as limits of front tracking approximations?
Fix \((\tau, \xi)\). Define \(U^\# = U^\#_{(\tau, \xi)}\) as the solution of the Riemann problem

\[
\begin{align*}
\frac{\partial w}{\partial t} + f(w) \frac{\partial w}{\partial x} &= 0, \\
w(\tau, x) &= \begin{cases} 
    u^+ = u(\tau, \xi^+) & \text{if } x > \xi \\
    u^- = u(\tau, \xi^-) & \text{if } x < \xi
\end{cases}
\end{align*}
\]

Then for every \(\hat{\lambda} > 0\) we expect

\[
\lim_{h \to 0^+} \frac{1}{h} \int_{\xi - h\hat{\lambda}}^{\xi + h\hat{\lambda}} \left| u(\tau + h, x) - U^\#_{(\tau, \xi)}(\tau + h, x) \right| \, dx = 0 \quad (E1)
\]
2. Comparison with solutions to a linear hyperbolic problem

Fix \((\tau, \xi)\). Choose \(\hat{\lambda} > 0\) larger than all wave speeds.

Define \(U^b = U^b_{(\tau, \xi)}\) as the solution of the linear Cauchy problem

\[
\begin{align*}
    w_t + \tilde{A}w_x &= 0 \\
    w(\tau, x) &= u(\tau, x)
\end{align*}
\]

with “frozen” coefficients: \(\tilde{A} = A(u(\tau, \xi))\)

Then, for \(a < \xi < b\) and \(h > 0\) we expect

\[
\frac{1}{h} \int_{a+\hat{\lambda}h}^{b-\hat{\lambda}h} \left| u(\tau + h, x) - U^b(\tau + h, x) \right| \, dx = O(1) \cdot \left( \text{Tot.Var.} \{ u(\tau, \cdot); [a, b] \} \right)^2
\]

(E2)
Theorem (A. Bressan, 1994)

Let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous w.r.t. the $L^1$ distance. Then $u$ is a weak solution to the system of conservation laws

$$u_t + f(u)_x = 0$$

obtained as limit of front tracking approximations if and only if the estimates (E1)-(E2) are satisfied for a.e. $\tau \in [0, T]$, at every $\xi \in \mathbb{R}$.

$$(E1) + (E2) \implies \lim_{h \to 0^+} \frac{\|u(\tau + h) - S_h u(\tau)\|_{L^1}}{h} = 0$$

Hence, by error estimate: $u(t) = S_t u(0)$
Proof. Insert points $x_i$ such that

\[ \text{Tot.Var.} \left\{ u(\tau); \ [x_{i-1}, x_i] \right\} < \varepsilon \]

\[
\frac{1}{h} \int_{-\infty}^{\infty} \left| u(\tau + h, x) - S_h u(\tau)(x) \right| \, dx = \sum_i \frac{1}{h} \int_{x_{i-1} - \lambda h}^{x_i + \lambda h} \cdots \, dx + \sum_i \frac{1}{h} \int_{x_{i-1} - \lambda h}^{x_i + \lambda h} \cdots \, dx 
\]

\[ = \sum_i A_i + \sum_i B_i \]

Estimate (E1) implies $A_i \to 0$ as $h \to 0$

Estimate (E2) implies $B_i \leq \varepsilon \cdot \text{Tot.Var.} \left\{ u(\tau); \ [x_{i-1}, x_i] \right\}$

\[ \sum_i B_i \leq \varepsilon \cdot \text{Tot.Var.} \left\{ u(\tau); \ \mathbb{R} \right\} \]
Uniqueness of weak solutions

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

- introduce a suitable set of admissibility + regularity assumptions
- show that these assumptions imply the estimates \((E1)\)\(+(E2)\)

\[ \Rightarrow u(t) = S_t \bar{u} \quad \text{for all} \quad t \geq 0 \]
A set of assumptions

(A1) (Conservation Equations)

$u : [0, T] \mapsto \mathcal{D}$ is continuous w.r.t. the $L^1$ distance.

The initial condition $u(0, x) = \bar{u}(x)$ holds.

Moreover, $u$ is a weak solution:

$$\int \int \left\{ u \varphi_t + f(u) \varphi_x \right\} \, dx \, dt = 0 \quad \text{for all } \varphi \in C^1_c$$

(A2) (Admissibility Conditions)

$u$ satisfies the Lax admissibility conditions at each point $(\tau, \xi)$ of approximate jump.
(A3) **(Tame Oscillation Condition)** For some constants $C$, $\hat{\lambda}$ the following holds. For every point $x \in \mathbb{R}$ and every $t, h > 0$ one has

$$|u(t + h, x) - u(t, x)| \leq C \cdot \text{Tot.Var.}\left\{ u(t, \cdot); \ [x - \hat{\lambda}h, x + \hat{\lambda}h] \right\}$$

(A4) **(Bounded Variation Condition)** There exists $\delta > 0$ such that, for every space-like curve $\{ t = \tau(x) \}$ with $|d\tau/dx| \leq \delta$ a.e., the function $x \mapsto u(\tau(x), x)$ has locally bounded variation.
Uniqueness results

\[ u_t + f(u)_x = 0 \quad \quad u(0, x) = \bar{u}(x) \]

**Theorem.** Let the system be strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear.

- Every weak solution \( u = u(t, x) \) obtained as limit of front tracking approximations satisfies all conditions (A1)–(A4).
- If \( u : [0, T] \mapsto \mathcal{D} \) satisfies (A1),(A2),(A3), then \( u(t) = S_t \bar{u} \) for \( t \geq 0 \). The same is true if \( u \) satisfies (A1), (A2), (A4).

(A.B. & P. LeFloch, 1997)

(A. B. & P. Goatin, 1999)

(A.B. & M. Lewicka, 2000)
Vanishing viscosity approximations

Goal: show that the weak solutions of the hyperbolic system

$$u_t + f(u)_x = 0$$

which satisfy suitable *admissibility conditions*

are precisely the limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$$

letting the viscosity $\varepsilon \to 0+$
Three basic cases: 1 - Smooth solutions

Let $u = u(t, x)$ be a smooth solution of the hyperbolic system

$$u_t + A(u)u_x = 0 \quad u(0, x) = \bar{u}(x)$$

Then a standard perturbation argument yields: $u = \lim_{\varepsilon \to 0^+} u^{\varepsilon}$, where

$$u_t^{\varepsilon} + A(u^{\varepsilon})u^{\varepsilon}_x = \varepsilon u^{\varepsilon}_{xx} \quad u^{\varepsilon}(0, x) = \bar{u}(x)$$

Remark. The conclusion is far from obvious if $u$ contains jumps.
2. Scalar conservation laws

\[ u_\varepsilon^t + f'(u_\varepsilon)u_\varepsilon^x = \varepsilon u_\varepsilon^{xx} \]

Total variation of viscous solutions is non-increasing in time

\[ (u_\varepsilon^x)_t + [f'(u_\varepsilon)u_\varepsilon^x]_x = \varepsilon (u_\varepsilon^x)_{xx} \]

Uniform BV bound \( \Rightarrow \) a subsequence converges strongly in \( L^1_{loc} \)

(A. I. Volpert 1968, S. Kruzhkov 1970)
3 - Linear hyperbolic systems

\[ u_t + A u_x = 0 \quad u(0, x) = \bar{u}(x) \]

\[ u_t^\varepsilon + A u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \quad u(0, x) = \bar{u}(x) \]

\[ l_i A = \lambda_i l_i \quad Ar_i = \lambda_i r_i \quad i = 1, \ldots, n \]

Solve componentwise: \( u_i = l_i \cdot u \)

\[ (u_i)_t + \lambda_i (u_i)_x = 0 \quad u_i(0, x) = l_i \cdot \bar{u}(x) \]

= limit of scalar viscous approximations \( u_i^\varepsilon = l_i \cdot u^\varepsilon \)

\[ (u_i^\varepsilon)_t + \lambda_i (u_i^\varepsilon)_x = \varepsilon (u_i^\varepsilon)_{xx} \quad u_i(0, x) = l_i \cdot \bar{u}(x) \]
The general case


Consider a strictly hyperbolic system with viscosity

\[ u_t + A(u)u_x = \varepsilon u_{xx} \quad u(0, x) = \bar{u}(x). \]  

\((CP)\)

If Tot.Var.\{\bar{u}\} is sufficiently small, then (CP) admits a unique solution  \( u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u} \), defined for all \( t \geq 0 \). Moreover, we have the estimates

\[ \text{Tot.Var.}\{S_t^\varepsilon \bar{u}\} \leq C \text{Tot.Var.}\{\bar{u}\}, \]  

\(\text{(BV bounds)}\)

\[
\left\| S_t^\varepsilon \bar{u} - S_t^\varepsilon \bar{v} \right\|_{L^1} \leq L \left\| \bar{u} - \bar{v} \right\|_{L^1}
\]

\(\text{(L}^1\text{ stability)}\)

(Convergence) As \( \varepsilon \to 0 \), the solutions \( u^\varepsilon \) converge to the trajectories of a semigroup \( S \) such that

\[
\left\| S_t \bar{u} - S_t \bar{v} \right\|_{L^1} \leq L \left\| \bar{u} - \bar{v} \right\|_{L^1} \quad t \geq 0.
\]
These vanishing viscosity limits can be regarded as the unique viscosity solutions of the hyperbolic Cauchy problem

\[ u_t + A(u)u_x = 0 \quad u(0, x) = \bar{u}(x). \]

In the conservative case \( A(u) = Df(u) \), the viscosity solutions are weak solutions of

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

satisfying the Liu entropy conditions.