Hyperbolic Systems of Conservation Laws

II - The Cauchy Problem

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Global in Time Solutions to the Cauchy Problem

\[
\begin{align*}
    u_t + f(u)_x &= 0 \\
    u(0,x) &= \bar{u}(x)
\end{align*}
\]

- Construct a sequence of approximate solutions \((u_m)_{m \geq 1}\)
- Show that (a subsequence) converges: \(u_m \rightarrow u\) in \(L^1_{loc}\)

\[
\implies u \text{ is a weak solution}
\]

Need: a-priori bound on the total variation (J. Glimm, 1965)
Building block: the Riemann Problem

\[ u_t + f(u)_x = 0 \]
\[ u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \]

- **B. Riemann 1860**: $2 \times 2$ system of isentropic gas dynamics
- **P. Lax 1957**: $n \times n$ systems (+ special assumptions)
- **T. P. Liu 1975**: $n \times n$ systems (generic case)
- **S. Bianchini 2003**: (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)
Solution to the Riemann problem

\[ \dot{\omega}_0 = u^- \quad \omega_0 = u^- \quad \dot{\omega}_3 = u^+ \quad \omega_3 = u^+ \]

is invariant w.r.t. rescaling symmetry:
\[ u^\theta(t, x) \equiv u(\theta t, \theta x) \quad \theta > 0 \]

describes local behavior of BV solutions near each point \((t_0, x_0)\)

describes large-time asymptotics as \(t \to +\infty\) (for small total variation)
Riemann Problem for Linear Systems

\[ u_t + Au_x = 0 \]

\[ u(0, x) = \begin{cases} 
  u^- & \text{if } x < 0 \\
  u^+ & \text{if } x > 0 
\end{cases} \]

\[ x / t = \lambda_1 \]

\[ x / t = \lambda_2 \]

\[ x / t = \lambda_3 \]

\[ \omega_0 = u^- \]

\[ \omega_1 \]

\[ \omega_2 \]

\[ \omega_3 = u^+ \]

\[ u^+ - u^- = \sum_{j=1}^{n} c_j r_j \]  

(sum of eigenvectors of \( A \))

intermediate states : \( \omega_i \equiv u^- + \sum_{j \leq i} c_j r_j \)

\[ i\text{-th jump: } \omega_i - \omega_{i-1} = c_i r_i \]  

travels with speed \( \lambda_i \)
Scalar Conservation Law

\[ u_t + f(u)_x = 0 \quad u \in \mathbb{R} \]

CASE 1: Linear flux: \( f(u) = \lambda u \).

Jump travels with speed \( \lambda \) \(\text{(contact discontinuity)}\)

\[ f'(u) = \lambda \]

\[ \begin{array}{c}
\text{u}^- \\
\lambda t \\
\text{u}^+ \\
\text{u(t)}
\end{array} \]
CASE 2: the flux $f$ is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \implies \text{centered rarefaction wave}$

$u^+ < u^- \implies \text{stable shock}$

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$
A class of nonlinear hyperbolic systems

\[ u_t + f(u)_x = 0 \]

\[ A(u) = Df(u) \quad A(u)r_i(u) = \lambda_i(u)r_i(u) \]

**Assumption (H) (P.Lax, 1957):** Each \( i \)-th characteristic field is

- either **genuinely nonlinear**, so that \( \nabla \lambda_i \cdot r_i > 0 \) for all \( u \)
- or **linearly degenerate**, so that \( \nabla \lambda_i \cdot r_i = 0 \) for all \( u \)
genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors $r_i$

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors $r_i$
Shock and Rarefaction curves

\[ u_t + f(u)_x = 0 \quad A(u) = Df(u) \]

**i-rarefaction** curve through \( u_0 \): \( \sigma \mapsto R_i(\sigma)(u_0) \)

= integral curve of the field of eigenvectors \( r_i \) through \( u_0 \)

\[ \frac{du}{d\sigma} = r_i(u), \quad u(0) = u_0 \]

**i-shock** curve through \( u_0 \): \( \sigma \mapsto S_i(\sigma)(u_0) \)

= set of points \( u \) connected to \( u_0 \) by an \( i \)-shock, so that \( u - u_0 \) is an \( i \)-eigenvector of the averaged matrix \( A(u, u_0) \)
\[ u_t + f(u)_x = 0 \quad u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \]

**CASE 1 (Centered rarefaction wave).** Let the \( i \)-th field be genuinely nonlinear.

If \( u^+ = R_i(\sigma)(u^-) \) for some \( \sigma > 0 \), then

\[
u(t, x) = \begin{cases} u^- & \text{if } x < t \lambda_i(u^-), \\ R_i(s)(u^-) & \text{if } x = t \lambda_i(s) \quad s \in [0, \sigma] \\ u^+ & \text{if } x > t \lambda_i(u^+) \end{cases}
\]

is a weak solution of the Riemann problem.
A centered rarefaction wave

\[ u + u(t) \]

\[ x = \lambda_i(u^-) \]

\[ x = \lambda_i(u^+) \]
CASE 2 (Shock or contact discontinuity). Assume that

\[ u^+ = S_i(\sigma)(u^-) \]

for some \( i, \sigma \). Let \( \lambda = \lambda_i(u^-, u^+) \) be the shock speed.

Then the function

\[
u(t, x) = \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t, \end{cases}
\]

is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff \( \sigma < 0 \).
Solution to a 2 x 2 Riemann problem

\[ \begin{align*}
2 & - u_1 \\
& + \omega_1 \\
R_1 & \omega_1 \\
R_2 & S_2 \\
1 - \text{rarefaction} & \\
0 & u = u_1 \\
& u = u^- \\
& u = u^+ \\
2 - \text{shock} &
\end{align*} \]
Solution of the general Riemann problem (P. Lax, 1957)

\[ u_t + f(u)_x = 0 \]
\[ u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases} \]

**Problem:** Find states \( u^- = \omega_0, \omega_1, \ldots, \omega_n \) such that

\[ \omega_0 = u^- \quad \omega_n = u^+ \]

and every couple \( \omega_{i-1}, \omega_i \) are connected by an elementary wave (shock or rarefaction)

\[ \begin{cases} \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases} \]
define: \( \Psi_i(\sigma)(u) = \begin{cases} 
R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\
S_i(\sigma)(u) & \text{if } \sigma < 0 
\end{cases} \)

\[
(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)
\]

Jacobian matrix at the origin: 
\[
J \doteq \begin{pmatrix} 
r_1(u^-) & r_2(u^-) & \cdots & r_n(u^-) 
\end{pmatrix}
\]
always has full rank

If \( |u^+ - u^-| \) is small, then the implicit function theorem yields existence and uniqueness of the intermediate states \( \omega_0, \omega_1, \ldots, \omega_n \)
General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)

\[ \omega_0 = u^- \]

\[ \omega_1 \]

\[ \omega_2 \]

\[ \omega_3 = u^+ \]
Global solution to the Cauchy problem

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]

**Theorem (Glimm, 1965).**

Assume:
- **system is strictly hyperbolic**
- **each characteristic field is either linearly degenerate or genuinely nonlinear**

Then there exists a constant \( \delta > 0 \) such that, for every initial condition \( \bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n) \) with
\[
\text{Tot.Var.}(\bar{u}) \leq \delta,
\]
the Cauchy problem has an entropy admissible weak solution \( u = u(t, x) \) defined for all \( t \geq 0 \).
Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in $t$-$x$ plane (Glimm scheme)

- at points where fronts interact (front tracking)
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans

![Diagram showing piecewise constant rarefaction fans](image-url)
Approximate the initial data $\bar{u}$ with a piecewise constant function.

Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$.

At each time $t_j$ where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem . . .

**NEED TO CHECK:**

- Total variation remains small
- Number of wave fronts remains finite
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves $\sigma', \sigma''$

Incoming: a $j$-wave of strength $\sigma'$ and an $i$-wave of strength $\sigma''$

Outgoing: waves of strengths $\sigma_1, \ldots, \sigma_n$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$
Incoming: two $i$-waves of strengths $\sigma'$ and $\sigma''$

Outgoing: waves of strengths $\sigma_1, \ldots, \sigma_n$. Then

$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| = O(1) \cdot |\sigma'\sigma''|\left(|\sigma'| + |\sigma''|\right)$$
Glimm functionals

Total strength of waves: \[ V(t) \doteq \sum_{\alpha} |\sigma_\alpha| \]

Wave interaction potential: \[ Q(t) \doteq \sum_{(\alpha,\beta) \in A} |\sigma_\alpha \sigma_\beta| \]

\( A \) \doteq couples of *approaching* wave fronts
Changes in $V, Q$ at time $\tau$ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) = O(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) = -|\sigma_\alpha \sigma_\beta| + O(1) \cdot V(\tau-) |\sigma_\alpha \sigma_\beta|$$

Choosing a constant $C_0$ large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

is nonincreasing, as long as $V$ remains small

Total variation initially small $\implies$ global BV bounds

$$\text{Tot.Var.}\{u(t, \cdot)\} \leq V(t) \leq V(0) + C_0 Q(0)$$

Front tracking approximations can be constructed for all $t \geq 0$
Keeping finite the number of wave fronts

At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts.

The total number of fronts can become infinite in finite time.

Need: a **Simplified Riemann Solver**, producing only one "non-physical" front.
A sequence of approximate solutions

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

\((u_\nu)_{\nu \geq 1}\) sequence of approximate front tracking solutions

- initial data satisfy \(\|u_\nu(0, \cdot) - \bar{u}\|_{L^1} \leq \varepsilon_\nu \to 0\)
- all shock fronts in \(u_\nu\) are entropy-admissible
- each rarefaction front in \(u_\nu\) has strength \(\leq \varepsilon_\nu\)
- at each time \(t \geq 0\), the total strength of all non-physical fronts in \(u_\nu(t, \cdot)\) is \(\leq \varepsilon_\nu\)
Existence of a convergent subsequence

\[ \text{Tot.Var.}\{u_\nu(t, \cdot)\} \leq C \]

\[ \|u_\nu(t) - u_\nu(s)\|_{L^1} \leq (t - s) \cdot \text{[total strength of all wave fronts]} \cdot \text{[maximum speed]} \]

\[ \leq L \cdot (t - s) \]

Helly’s compactness theorem \(\implies\) a subsequence converges

\[ u_\nu \rightarrow u \quad \text{in} \quad L^1_{loc} \]
Claim: \( u = \lim_{\nu \to \infty} u_\nu \) is a weak solution

\[
\int \int \left\{ \phi_t u + \phi_x f(u) \right\} \, dx dt = 0 \quad \phi \in C^1_c \left( ]0, \infty[ \times \mathbb{R} \right)
\]

Need to show:

\[
\lim_{\nu \to \infty} \int \int \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} \, dx dt = 0
\]
Assume $\phi(t, x) = 0$ outside the strip $[0, T] \times \mathbb{R}$. Define

$$
\Delta u_\nu(t, x_\alpha) \doteq u_\nu(t, x_\alpha^+) - u_\nu(t, x_\alpha^-)
$$

$$
\Delta f(u_\nu(t, x_\alpha)) \doteq f(u_\nu(t, x_\alpha^+)) - f(u_\nu(t, x_\alpha^-))
$$

$$
\Phi_\nu \doteq (\phi \cdot u_\nu, \phi \cdot f(u_\nu)).
$$

Use the divergence theorem on each polygonal domain $\Gamma_j$ where $u_\nu$ is constant:

$$
\sum_j \int\int_{\Gamma_j} \text{div } \Phi_\nu(t, x) \, dx \, dt = \sum_j \int_{\partial \Gamma_j} \Phi_\nu \cdot n \, d\sigma
$$
\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} \left\{ \phi_t(t, x) u_\nu(t, x) + \phi_x(t, x) f(u_\nu(t, x)) \right\} \, dx \, dt \\
= \sum_{j} \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma
\]

\[
\limsup_{\nu \to \infty} \left| \sum_{j} \int_{\partial \Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma \right|
\leq \limsup_{\nu \to \infty} \left| \sum_{\alpha \in S \cup R \cup N^P} \left[ \dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha)) \right] \phi(t, x_\alpha(t)) \right|
\leq \left( \max_{t, x} |\phi(t, x)| \right) \cdot \limsup_{\nu \to \infty} \left\{ O(1) \cdot \sum_{\alpha \in R} \varepsilon_\nu |\sigma_\alpha| + O(1) \cdot \sum_{\alpha \in N^P} |\sigma_\alpha| \right\}
= 0
The Glimm scheme

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

Assume: all characteristic speeds satisfy \( \lambda_i(u) \in [0, 1] \)

This is not restrictive. If \( \lambda_i(u) \in [-M, M] \), simply change coordinates:

\[ y = x + Mt, \quad \tau = 2Mt \]

Choose:

- a grid in the \( t-x \) plane with step size \( \Delta t = \Delta x \)
- a sequence of numbers \( \theta_1, \theta_2, \theta_3, \ldots \) uniformly distributed over \([0, 1]\)

\[
\lim_{N \to \infty} \frac{\#\{j : 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \}}{N} = \lambda \quad \text{for each } \lambda \in [0, 1].
\]
Glimm approximations

Grid points: \( x_j = j \cdot \Delta x, \quad t_k = k \cdot \Delta t \)

- for each \( k \geq 0 \), \( u(t_k, \cdot) \) is piecewise constant, with jumps at the points \( x_j \). The Riemann problems are solved exactly, for \( t_k \leq t < t_{k+1} \)

- at time \( t_{k+1} \) the solution is again approximated by a piecewise constant function, by a sampling technique.
Example: Glimm’s scheme applied to a solution containing a single shock

\[ U(t, x) = \begin{cases} 
  u^+ & \text{if } x > \lambda t \\
  u^- & \text{if } x < \lambda t 
\end{cases} \]

Fix \( T > 0 \), take \( \Delta x = \Delta t = T/N \)

\[ x(T) = \# \{ j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \} \cdot \Delta t \]

\[ = \frac{\# \{ j ; \ 1 \leq j \leq N, \ \theta_j \in [0, \lambda] \}}{N} \cdot T \rightarrow \lambda T \quad \text{as } N \rightarrow \infty \]
Random sampling at points determined by the equidistributed sequence \((\theta_k)_{k \geq 1}\)

\[
\lim_{N \to \infty} \frac{\#\{j \; : \; 1 \leq j \leq N, \; \theta_j \in [0, \lambda]\}}{N} = \lambda \quad \text{for each } \lambda \in [0, 1].
\]

Need fast convergence to uniform distribution. Achieved by choosing:

\[
\theta_1 = 0.1, \; \ldots, \; \theta_{759} = 0.957, \; \ldots, \; \theta_{39022} = 0.22093, \; \ldots
\]

**Convergence rate:**

\[
\lim_{\Delta x \to 0} \frac{\|u^{\text{Glimm}}(T, \cdot) - u^{\text{exact}}(T, \cdot)\|_{L^1}}{\sqrt{\Delta x} \cdot |\ln \Delta x|} = 0
\]

(A.Bressan & A.Marson, 1998)