1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Consider the hyperbolic equation
\[
\begin{aligned}
  u_{tt} - \Delta u &= c(x) u + g(u) \quad x \in \Omega, \\
  u &= 0 \quad x \in \partial \Omega.
\end{aligned}
\] (1)

Construct a function $\phi(x, u)$ such that, for every smooth solution of (1), the total energy
\[
E = \int_{\Omega} \left( \frac{|u_t|^2}{2} + \frac{|
abla u|^2}{2} + \phi(x, u) \right) \, dx
\]
is constant in time. (Hint: write an identity that should be satisfied by the partial derivative $\phi_u$, in order that $\frac{dE(t)}{dt} = 0$. )

2. On the space $L^1(\mathbb{R})$, consider the semigroup $\{S_t ; t \geq 0\}$ generated by the operator $Au = u_{xx}$.

(i) Show that this is a contractive semigroup on the space $L^1(\mathbb{R})$.

(ii) Using Duhamel’s formula, write out the mild solution to the Cauchy problem
\[
\begin{aligned}
  u_t &= u_{xx} + \frac{t}{1 + x^2}, \\
  u(0, x) &= e^{-|x|}.
\end{aligned}
\] (2)

(iii) Write an integral equation satisfied by the mild solution of the semilinear Cauchy problem
\[
\begin{aligned}
  u_t &= u_{xx} + \sin u, \\
  u(0, x) &= f(x).
\end{aligned}
\] (3)

Explain why this equation has a unique solution, defined for all $t \geq 0$.

3. Decide which of the following maps is Lipschitz continuous from $L^1(\mathbb{R})$ into itself.
\[
\begin{aligned}
  (\Lambda_1 u)(x) &= \sin u(x), & (\Lambda_2 u)(x) &= \cos u(x), \\
  (\Lambda_3 u)(x) &= \frac{\int_{-\infty}^{\infty} u(y) \, dy}{1 + x^2}, & (\Lambda_4 u)(x) &= u^2(x).
\end{aligned}
\]

In the positive case, determine the Lipschitz constant. In other words, find $C$ such that
\[
\|\Lambda u - \Lambda v\|_{L^1} \leq C \cdot \|u - v\|_{L^1}.
\]
1. On the open interval $\Omega = [0, 3[$, consider the boundary value problem

\[
\begin{cases}
-\frac{d^2u}{dx^2} = 1 & 0 < x < 3, \\
u(0) = u(3) = 0.
\end{cases}
\]  

(1)

Consider the two linearly independent functions $\varphi_1, \varphi_2 \in H^1_0([0, 3[)$, defined by

\[
\varphi_1(x) = \begin{cases}
x & \text{if } x \in [0, 1[ \\
2 - x & \text{if } x \in [1, 2[ \\
0 & \text{if } x \in [2, 3[.
\end{cases}
\]
\[
\varphi_2(x) = \begin{cases}
x - 1 & \text{if } x \in [1, 2[ \\
3 - x & \text{if } x \in [2, 3[.
\end{cases}
\]

Explicitly compute the Galerkin approximation $U(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$ such that

\[
B[U, \varphi_i] = \int_0^3 U_x \cdot \varphi_{i,x} \, dx = \int_0^3 1 \cdot \varphi_i \, dx = (1, \varphi_i)_{L^2} \quad i = 1, 2.
\]

Compare $U$ with the exact solution of (1).

2. On the open interval $\Omega = [0, 3[$, consider the parabolic problem

\[
\begin{cases}
\frac{du}{dt} = \frac{d^2u}{dx^2} & 0 < x < 3, \\
u(t, 0) = u(t, 3) = 0, \\
u(0, x) = 1.
\end{cases}
\]

(2)

Explicitly compute the Galerkin approximation $U(t, x) = c_1(t) \varphi_1(x) + c_2(t) \varphi_2(x)$, choosing the functions $c_1, c_2$ so that

\[
(U(0, \cdot) - 1, \varphi_i)_{L^2} = 0 \quad i = 1, 2, \quad t = 0,
\]
\[
(U_t, \varphi_i)_{L^2} + B[U, \varphi_i] = 0 \quad i = 1, 2, \quad t > 0.
\]

Compare $U$ with the exact solution of (2).

3. Let $\Omega = \{(x, y) ; \ x_1^2 + x_2^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, and let $u$ be a smooth solution to the equation

\[
u_{tt} = 2ux_1x_1 + x_2u_{x_1}x_2 + 3ux_2x_2 + \frac{1}{2}u_{x_1} \quad \text{on } \Omega \times [0, T],
\]
\[
u = 0 \quad \text{on } \partial \Omega \times [0, T].
\]

(i) Write the equation (3) in the form $u_{tt} + Lu = 0$, showing that the operator $L$ is symmetric (i.e., $a_{12} = a_{21}$) and uniformly elliptic on the domain $\Omega$.

(ii) Define a suitable energy $e(t) = \text{[kinetic energy]} + \text{[elastic potential energy]}$, and check that it is constant in time.
1. Consider the space of square summable sequences of real numbers:

\[ X = \left\{ x = (x_1, x_2, x_3, \ldots) ; \quad \| x \| = \left( \sum_k |x_k|^2 \right)^{1/2} < \infty \right\} \]

(i) Show that the linear operator \( Ax = ( -x_1, -2x_2, -3x_3, \ldots ) \) is not bounded on \( X \).

(ii) Explicitly write out the semigroup generated by \( A \). In other words, for every \( x \in X \), compute \( S_t x \). Prove that this semigroup is contractive.

(iii) Prove that, for every \( x \in X \) and \( t > 0 \) one has \( S_t x \in \text{Dom}(A) \), even if \( x \notin \text{Dom}(A) \). Show that \( \| AS_t \| \leq \sup_{\lambda > 0} \lambda e^{-t\lambda} < \infty \).

2. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. A function \( u \in \text{H}_0^2(\Omega) \) is a weak solution of the biharmonic equation

\[
\begin{align*}
\Delta^2 u &= f \quad x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad x \in \partial \Omega,
\end{align*}
\]

if

\[
\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in \text{H}_0^2(\Omega).
\]

Given \( f \in \text{L}^2(\Omega) \), prove that the boundary value problem (1) has a unique weak solution.

Hint: show that the bilinear form \( B[u, v] \) on \( \text{H}_0^2(\Omega) \) defined by the left hand side of (2) is strictly positive definite on \( \text{H}_0^2(\Omega) \). Notice that, if \( u \in \mathcal{C}_c^\infty(\Omega) \), integrating by parts one obtains

\[
\int_\Omega u_{x,x} u_{x,x} \, dx = \int_\Omega u_{x,x} u_{x,x} \, dx \leq \frac{1}{2} \int_\Omega (u_{x,x}^2 + u_{x,x}^2) \, dx.
\]
1. Let $A$ be the generator of a contractive semigroup, on a Banach space $X$. Show that, for every $\lambda > 0$, the bounded linear operator $A_\lambda = \lambda A (\lambda I - A)^{-1}$ also generates a contractive semigroup.

2. On the space $L^1(\mathbb{R})$, consider the linear operators $S_t$ defined by 
   
   $$(S_t f)(x) = e^{-2t} f(x + t).$$
   
   (i) Prove that the family of linear operators $\{S_t; \ t \geq 0\}$ is a strongly continuous, contractive semigroup on $L^1(\mathbb{R})$. Find the generator $A$ of this semigroup. What is $Dom(A)$?
   
   (ii) Show that the family of operators $\{S_t; \ t \geq 0\}$ is NOT a strongly continuous semigroup on $L^\infty(\mathbb{R})$.

3. Let $\{S_t; \ t \geq 0\}$ be a strongly continuous semigroup of linear operators on $\mathbb{R}^n$. Prove that there exists an $n \times n$ matrix $A$ such that $S_t = e^{tA}$ for every $t \geq 0$.

4. On the space $L^1(\mathbb{R})$, consider the operator $Au = \frac{\partial}{\partial x} u$ with domain 
   
   $Dom(A) = \left\{ u \in L^1(\mathbb{R}); \ u \ is \ absolutely \ continuous, \ u_x \in L^1(\mathbb{R}) \right\}$.
   
   (i) Describe the semigroup $\{S_t; t \geq 0\}$ generated by $A$.
   
   (ii) For any $u \in L^1(\mathbb{R})$ and $h > 0$, construct the backward Euler approximation $E^- h u$.

5 (extra credit) Fix a time $T > 0$. On the space $X = L^1([0,1])$, construct a strongly continuous, contractive semigroup of linear operators $\{S_t; t \geq 0\}$ such that $\|S_t\| = 1$ for $0 \leq t < T$ but $S_t = 0$ for $t \geq T$. 
1. On the space \( L^2([0, \infty[) \), consider the linear operator

\[
Lu(x) = u(x + 1).
\]

(i) Find the adjoint operator \( L^* \), such that \( (u, L^*v)_{L^2} = (Lu, v)_{L^2} \) for every \( u, v \in L^2([0, \infty[) \).

(ii) Find the kernel and the range of \( L \) and \( L^* \). Do the two kernels have the same dimension?

2. Assume \( f \in L^2([0, \pi]) \). Show that the problem

\[
\begin{align*}
  u'' + 9u &= f, & u(0) &= u(\pi) = 0
\end{align*}
\]

has a solution if and only if

\[
\int_0^\pi f(x) \sin 3x \, dx = 0.
\]

Is the solution unique?

3. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and consider the operator

\[
Lu = -\sum_{ij} (a^{ij}(x)u_{x_i})_{x_j} + \sum_i b^i u_{x_i} + c(x)u,
\]

where the coefficients \( b^i \) are constant. Assume that \( a^{ij}, c \in L^\infty(\Omega) \), with \( c(x) \geq 0 \), and that \( L \) is uniformly elliptic, so that \( \sum_{ij} a^{ij} \xi_i \xi_j \geq \theta \sum_i \xi_i^2 \), for some \( \theta > 0 \).

(i) Write the corresponding bilinear form \( B[u,v] \). Prove that it is strictly positive definite on \( H^1_0(\Omega) \). (Hint: compute the divergence of the vector \( (b^1, \ldots, b^n) \frac{u^2}{2} \))

(ii) Prove that, for every \( f \in L^2(\Omega) \), the problem

\[
\begin{align*}
  \begin{cases}
    Lu = f & x \in \Omega, \\
    u = 0 & x \in \partial\Omega,
  \end{cases}
\end{align*}
\]

has a unique weak solution.
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Homework # 5 (due Thursday, 2/26/2015)

1. Let \( \Omega = \{(x_1, x_2); \ x_1^2 + x_2^2 < 1\} \) be the unit disc in \( \mathbb{R}^2 \). Consider the operator

\[
Lu = -(u_{x_1 x_1} + u_{x_2 x_2} + x_1 u_{x_1 x_2} + 3x_2^2 u_{x_1})
\]

(i) Check whether the operator \( L \) is elliptic on \( \Omega \).

(ii) Given \( f \in L^2(\Omega) \), explain what it means for a function \( u \) to be a weak solution of the equation

\[
\begin{cases}
Lu = f & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]

2. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and let \( a, c : \Omega \mapsto [1, 2] \) be smooth functions. Define \( H \) to be the Hilbert space \( H^1_0(\Omega) \) with the equivalent inner product

\[
(u, v)_H = \int_\Omega a(x) \nabla u(x) \cdot \nabla v(x) \, dx + \int_\Omega c(x) u(x) v(x) \, dx.
\]

Let \( \iota : H \mapsto L^2(\Omega) \) be the identity map, and let \( \iota^* : L^2(\Omega) \mapsto H \) be the adjoint operator.

Prove that, for every \( f \in L^2(\Omega) \), the function \( \iota^* f \in H \) provides the weak solution to an elliptic boundary value problem. Which equation does \( \iota^* f \) solve?

3. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and let \( u \) be a weak solution to

\[
\begin{cases}
-\Delta u + u = f & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]

(i) Prove that \( \|u\|_{L^2} \leq \|f\|_{L^2} \).

(ii) Prove that \( \|\nabla u\|_{L^2} \leq \|f\|_{L^2} \).

Hint: in both cases, use the definition of weak solution taking \( v = u \) as test function.
1. Let $\Omega$ be an open ball in $\mathbb{R}^n$. Show that there exists a constant $C$ such that

$$\int_{\Omega} u^2(x) \, dx \leq C \int_{\Omega} |\nabla u(x)|^2 \, dx + C \left| \int_{\Omega} u(x) \, dx \right|^2.$$

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $c(\cdot)$ be a continuous function on $\overline{\Omega}$.

(i) Show that there exists $\mu > 0$ such that, if $c(x) > -\mu$ for all $x$, then the bilinear operator

$$B[u, v] = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} c(x)u(x)v(x) \, dx$$

satisfies the assumption of the Lax-Milgram theorem on the space $H^1_0(\Omega) = W^{1,2}_0(\Omega)$.

(ii) By (i), for every $f \in L^2(\Omega)$, there exists $u \in H^1_0(\Omega)$ such that

$$B[u, v] = (f, v)_{L^2} \quad \text{for every } v \in H^1_0(\Omega).$$

What PDE + boundary conditions does $u$ solve, in a weak sense?

3. Consider the open interval $\Omega = ]-1, 1[$, and the Hilbert-Sobolev space $H^1_0(\Omega)$.

(i) Show that the Dirac measure $\delta_0 : v \mapsto v(0)$ (= a unit mass concentrated at the origin) is a continuous linear functional on $H^1_0(\Omega)$.

Is the Dirac measure an element of the space $H^1_0(\Omega)$?

(ii) Explicitly determine a function $u \in H^1_0(\Omega)$ such that

$$(u, v)_{H^1} = v(0) \quad \text{for every } v \in H^1_0(\Omega).$$

Hint: solve the boundary value problem

$$u - u'' = 0 \quad x \in ]-1, 0[ \cup ]0, 1[,$$

$$u(-1) = u(1) = 0, \quad u'(0-) - u'(0+) = 1.$$
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Homework # 3 (due Thursday, 2/12/2015)

1. Prove that, for any $1 \leq p < \infty$, one has $W^{1,p}(\mathbb{R}^n) = W^{1,p}_0(\mathbb{R}^n)$.

   Hint: Let $u \in W^{1,p}(\mathbb{R}^n)$. Given $\varepsilon > 0$, find $\tilde{u} \in C^\infty(\mathbb{R}^n)$ such that $\|\tilde{u} - u\|_{W^{1,p}} < \varepsilon$. Then take a $C^\infty$ function \( \psi : \mathbb{R} : [0,1] \), with \( \psi(r) = 1 \) if \( x \leq 0 \), and \( \psi(r) = 0 \) if \( r > 1 \).

   Show that the functions

   \[ u_k(x) = \tilde{u}(x) \cdot \psi(|x| - k) \]

   are $C^\infty$ with compact support. Prove that $\|u_k - \tilde{u}\|_{W^{1,p}} \to 0$ as $k \to \infty$.

2. Consider the sequence of functions $u_k(x) = \sin kx$ For a fixed $\varepsilon > 0$, define the regularization

   \[ u_k^\varepsilon(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} u_k(y) dy. \]

   (i) Prove that, for a fixed $\varepsilon > 0$, all functions $u_k^\varepsilon$, $k \geq 1$ are uniformly Lipschitz continuous.

   (ii) By (i), on the compact interval $[0,2\pi]$, by Ascoli’s theorem (possibly taking a subsequence), one has the uniform convergence $\lim_{k \to \infty} u_k^\varepsilon = \bar{u}^\varepsilon$. Describe the function $\bar{u}^\varepsilon$. What is $\bar{u} = \lim_{\varepsilon \to 0} \bar{u}^\varepsilon$ ?

   (iii) Estimate the limit

   \[ \lim_{k \to \infty} \int_0^{2\pi} |u_k^\varepsilon(x) - u_k(x)| \, dx. \]

   Does this approach zero as $\varepsilon \to 0$ ? Is it true that $\|u_k - \bar{u}\|_{L^1([0,\pi])} \to 0$ ?

3. (i) Let $\Omega \subset \mathbb{R}^2$ be an open ball. Assume that $f \in W^{1,p}(\Omega)$. For which values of $p,q$ can one conclude that the product $f \cdot D_{x_1} f$ lies in $L^q(\Omega)$ ?

   (ii) Answer the same question in the case where $\Omega$ is a ball in $\mathbb{R}^3$. 

\[ \]
Let $\xi : \mathbb{R} \mapsto \mathbb{R}$ be a $C^1$, strictly increasing function (with $\xi'(x) > 0$ for every $x$), mapping $\Omega' = ]a, b[ \text{ onto } \Omega = ]c, d[$. Let $f \in W^{1,p}(\Omega)$ and define the composite function

$$g(x) = f(\xi(x)).$$

Prove that $g \in W^{1,p}(\Omega')$, and compute the weak derivative of $g$.

2. Verify that if $n > 1$ the unbounded function

$$u = \log \log \left(1 + \frac{1}{|x|}\right)$$

is in the space $W^{1,n}(\Omega)$. Here $\Omega \subset \mathbb{R}^n$ is the unit ball centered at the origin.

This shows that functions $u \in W^{1,n}(\mathbb{R}^n)$ need not be continuous, or bounded.

3. Let $\Omega \subset \mathbb{R}^n$ be the unit ball centered at the origin. Consider the problem of minimizing the integral $\int_{\Omega} |\nabla u|^p \, dx$ among all continuous functions that vanish on $\partial \Omega$ and take the value 1 at the origin. Assuming that $u(x) = w(|x|)$ is radially symmetric, this leads to the standard problem in the Calculus of Variations:

$$\int_0^1 r^{n-1} |w'(r)|^p \, dr$$

subject to: $w(0) = 1, \; w(1) = 0.$ (1)

Study whether this problem has solutions, depending on $n, p$. Hint: consider the Euler-Lagrange equations

$$\frac{\partial L}{\partial w} = \frac{d}{dr} \frac{\partial L}{\partial w'}, \quad L(r, w, w') = r^{n-1} |w'|^p.$$ (2)

Prove that

(i) If $p > n$, the second order ODE (2) has a solution with boundary data (1)

(ii) If $p < n$, no solution exists. In this case, for any $\varepsilon > 0$ one can find a function $w$ satisfying (1), such that

$$\int_0^1 r^{n-1} |w'(r)|^p \, dr < \varepsilon.$$

As a consequence, the norm $\|u\|_{C^0(\Omega)} = \sup_{x \in \Omega} |u(x)|$ cannot be controlled by the norm $\|u\|_{W^{1,p}(\Omega)}$. 

9
1. Which of the following spaces \((X, \| \cdot \|)\) is a Banach space? Motivate your answers.

(i) \(X = \) set of all continuous functions on \([0, 1]\),
\[
\|u\| = \left| \int_0^1 u(x) \, dx \right|.
\]

(ii) \(X = \) set of all polynomials of degree \(\leq 3\),
\[
\|u\| = \int_0^1 |u(x)| \, dx.
\]

(iii) \(X = \) set of all convex functions on \([0, 1]\),
\[
\|u\| = \sup_{x \in [0, 1]} |u(x)|.
\]

(iv) \(X = \) set of all absolutely continuous functions on \([0, 1]\),
\[
\|u\| = |u(0)| + \int_0^1 |u'(x)| \, dx.
\]

2. On the open interval \((-1, 1]\), consider the function
\[
u(x) = x^2 \cdot \sin \frac{1}{x^4} \quad \text{for } x \neq 0, \quad \nu(0) = 0.
\]

- Does this function have a weak derivative on the open set \(\Omega = (0, \infty]\)?
- Does this function have a weak derivative on the open set \(\Omega = ]-1, 1]\)?
  (If yes, compute it. If no, explain why).

3. Let \(\Omega = ]a, b[\) be an open interval. Prove that (after a possible modification on a set of measure zero):

(i) \(W^{1, \infty}(]a, b[)\) is the space of all functions which are Lipschitz continuous on \([a, b[\)

(ii) \(W^{1,p}_0(]a, b[)\) is the space of absolutely continuous functions \(f : ]a, b[ \to \mathbb{R}\) with
\[
Df \in L^p(]a, b[), \quad \lim_{x \to a+} f(x) = \lim_{x \to b-} f(x) = 0
\]

(iii) For \(1 \leq p < \infty\) one has \(W^{0,p}_0(]a, b[) = W^{0,p}(]a, b[) = L^p(]a, b[).\)

(iv) (extra credit) What is \(W^{0,\infty}_0(]a, b[)\)?