1. Given $\hat{f}(\omega) = \mathcal{F}(f)$, show that $|\hat{f}(\omega)|^2$ (the power spectrum) is an even function of $\omega$ if $f(t)$ is real.

2. Keener 3.1.1 (p.128)

3. Show that the eigenvalues $\lambda$ of a Sturm-Liouville problem for $u(x)$,

$$Lu = (p(x)u')' + q(x)u = \lambda u, \quad x \in [a,b]$$

are real.

4. Keener 3.2.3 (b) (p.128)

5. Keener 3.4.2 (b) (p.129)

6. Keener 3.4.4 (p.129)

7. Consider the wave equation for the displacement $u(x,t)$ of a string with nonuniform density $\rho = \rho(x)$:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}, \quad x \in [0,\ell], \; t > 0$$

where $T_0$ is the tension in the string. The boundary conditions given are $u(0,t) = 0$ and $u(\ell,t) = 0$.

(a) Assuming a separation of variables solution $u(x,t) = f(x)g(t)$, show that $f$ obeys the Sturm-Liouville equation.

(b) In what function space are the eigenfunctions $f_n(x)$ orthogonal?

(c) From the equation for $f_n$, obtain an integral equation for the eigenvalues $\lambda_n$ in terms of $f_n$ and $\rho(x)$ (“the Rayleigh quotient”). Use this to show that $\lambda_n \geq 0$, $\forall n$. 