1. Consider the function $f(x)$ on $[-\pi, \pi]$, with orthogonal basis sets based on the $\sin kx$ and $\cos kx$.

(a) If $f(x)$ is odd, show that the projections along either the sines or the cosines are zero.

(b) If $f(x)$ is even, show that the projections along the other set are zero.

(c) Show that an arbitrary function $g(x)$ can be written as a sum of odd and even functions.

(d) How would all of this be relevant for functions defined on $[0, 2\pi]$?

(e) Given another orthogonal basis $\phi_i$, under what conditions on $\phi$ would the odd and even functions behave as they do in (a)-(b)?

2. Give two examples of a Banach space which is not a Hilbert space.

3. Assuming that the operations of summation and integration can be interchanged, show that for

   $$f = \sum \alpha_i \phi_i \quad \text{and} \quad g = \sum \beta_i \phi_i,$$

with normalized basis vectors, we have the generalized Parseval's equality:

   $$\int_a^b f(x)g(x)dx = \sum_{n=1}^{\infty} \alpha_n \beta_n$$

4. Given $f \in L^2[a, b]$, show that any orthonormal basis of $L^2$ defines a transformation from $L^2$ to $\ell^2$.

5. Given that

   $$f(x) = \begin{cases} x^2 + e^x, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$$

write the Fourier sine and cosine series representing $f$. Simplify but do not evaluate the coefficient integrals. (hint: use even and odd parts)